This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

NYPL RESEARCH LIBRARIES
3 3433 06908727 2

OFC

OFC

A

THEORETICAL

AND

PRACTICAL ARITHMETIC:

IN WHICH

THE PRINCIPLES OF THAT SCIENCE ARE CLEARLY AND FULLY EXPLAINED;

BEING INTENDED AS AN INTRODUCTION TO THE

Higher Branches of Mathematics.

BY BEZOUT.

ENLARGED, AND ADAPTED TO THE USE OF YOUNG TRADERS, BANKERS, &c. &c.

BY F. PEYRARD,

EX-PROFESSOR OF MATHEMATICS AND ASTRONOMY OF THE ROYAL BOURBON COLLEGE, &c. &c.

TRANSLATED FROM THE FRENCH, IMPROVED AND ADAPTED TO THE CURRENCY OF THE UNITED STATES.

BY NOBLE HEATH.

NEW-YORK:

PUBLISHED BY SAMUEL WOOD & SONS, 261 PEARL-STREET, AND BY SAMUEL S. WOOD & CO. 212 MARKET-STREET, BALTIMORE.

1825.

THE NEW YORK
PUBLIC LIBRARY
787159 A
ASTOR, LENOX AND
TILDEN FOUNDATIONS
R 1985 L

Southern District of New-York, ss.

BE IT REMEMBERED, That on the tenth day of August, A.D. 1825, in the fiftieth year of the Independence of the United States of America, NOBLE HEATH, of the said district, hath deposited in this office the title of a book, the right whereof he claims as Proprietor, in the words following, to wit—

"A Theoretical and Practical Arithmetic: in which the Principles of that Science are clearly and fully explained; being intended as an Introduction to the Higher Branches of Mathematics, By Bezout. Enlarged, and adapted to the use of young Traders, Bankers, &c. &c. By F. Peyrard, Ex-Professor of Mathematics and Astronomy of the Royal Bourbon College, &c. &c. Translated from the French; improved, and adapted to the Currency of the U. S. By Noble Heath."

In conformity to the Act of the Congress of the United States, entitled "An Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the times therein mentioned," and also to an Act, entitled "An Act, supplementary to an Act, entitled An Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the times therein mentioned, and extending the benefits thereof to the arts of designing, engraving, and etching historical and other Prints."

JAMES DILL,
Clerk of the Southern District of New-York.

PREFACE.

MATHEMATICAL Sciences, above all others, delight us by the ease and certainty with which they attain their object. He who habituates himself to a close examination of their fundamental principles, and of the mutual dependence and connexion of these principles, acquires a justness of conception, a conciseness of reasoning, and a strength of judgment, scarcely attainable by any other means.

As Arithmetic may justly be considered the basis of mathematical science, the knowledge of its principles is an acquisition of great importance, and books in which those principles are clearly and fully developed, form a desideratum the want of which has long subjected both teachers and students to much inconvenience;—the arithmetics in common use being generally an assemblage of rules and examples without demonstration. The study of such rules is uninteresting and wearisome; and the scholar, after having committed them to memory, and learned by their means, to perform arithmetical calculations, as it were, mechanically, quits the study entirely ignorant of the principles upon which they subsist; and if, as is often the case, he forgets these dogmas, he loses with them all the fruits of his labour.

On perusing the present work in the original, I was struck with the ease and perspicuity of the style, in which Bezout has developed the principles and demonstrated the different operations of arithmetic; and from a full persuasion that it would be a great relief to teachers, and amply reward the investigation of the industrious student, I determined upon its translation. In the execution of my design, I have endeavoured to keep as near to the original, as the genius of our language, and the adaptation of the work to the commerce of this country, would permit. The explanation of the signs made use of in the work, the elucidation of duodecimals, some practical examples, and the insertion of the table of logarithms from one to ten thousand, instead of a small table from one to two hundred, are the principal augmentations which I have thought proper to make. The text of Bezout is numbered according to the order which he established; and the parts marked with an asterisk, are the augmentations made by F. Peyrard, which form a very valuable addition to the elegant original.

THE TRANSLATOR.

New-York, 1825.

BEZOUT'S ARITHMETIC.

1. In general, every thing is called *quantity*, which is susceptible of increase and diminution. Extent, duration, weight, etc. are quantities. Every thing that is quantity is the object of the mathematics; but arithmetic, which makes a part of these sciences, only considers quantities when they are expressed in numbers.

2. Arithmetic is then the science of numbers: it considers their nature and properties; and its object is to give easy methods for representing numbers, and for compounding and decompounding them; which is called *calculating*.

3. To form an exact idea of numbers, we must first know what is understood by a unit.

4. The unit is a quantity taken, in general, arbitrarily, to serve as a term of comparison to all quantities of the same kind: thus when we say, a certain body weighs *five* pounds, the pound is the unit; it is the quantity to which we compare the weight of the body; we might, with equal propriety, take the ounce for the unit, and then the weight of the same body would be signified by eighty.

5. Number shows of how many units or parts of a unit a quantity is composed.

If the quantity be composed of whole units, the number which expresses it is called *a whole number;* and if it be composed of whole units and parts of a unit, or simply of parts of a unit, then the number is called *fractional* or *fraction: three and a half* is a fractional number; *three-fourths* is a fraction. Whole units are called integers.

6. A number which we express without signifying the species of the units, as when we say simply *three* or *three times*, *four* or *four times*, is called an *abstract number;* and when we express at the same time the species of the units, as when we say *four books*, *three dollars*, it is called a *concrete number*.

OF NUMERATION AND DECIMALS.

7. Numeration is the art of expressing all numbers by a limited number of names and characters: these characters are called *figures*.

8. The characters made use of in numeration, and the names of the numbers which they represent, are as follows:

cipher, or nought	unit, or one	two	three	four	five	six	seven	eight	nine
0	1	2	3	4	5	6	7	8	9

To represent all other numbers with these ten characters, ten units are expressed by a single unit, and we count tens as we count the simple units, that is to say, we count two tens, three tens, etc. as far as 9: to represent these new units, we employ the same figures as for the simple units, but we distinguish them by the place which we make them occupy, in placing them on the left of the simple units.

Thus, to represent *fifty-four*, which contains five tens and four units, we write 54.

To represent *sixty*, which contains an exact number of tens and no units, we write 60, putting a cipher in the place of units, which shows that there are no simple units, and makes the figure 6 represent a number of tens. We can, by this means, count as far as ninety-nine inclusive.

9. Let us remark as we go along, this property of numeration; namely, that a figure placed on the left of another figure, or of a cipher, represents a number ten times as great as if it were alone.

10. From 99 we count as far as *nine hundred and ninety-nine*, in a similar manner. Of ten tens, we compose a single unit, which we call *hundred;* we count hundreds from one to nine, and represent them by the same figures, but in placing these figures on the left of the tens.

Thus, to express *eight hundred and fifty-nine*, which number contains eight hundreds, five tens, and nine units, we write 859. If it were *eight hundred and nine*, which contains eight hundreds, no tens, and nine units, we should write 809; that is to say, we should put a cipher in the place of the tens which are wanting. If the units were also wanting, we

should put two ciphers: thus to express *eight hundred*, we write 800.

11. Let us again remark, that a figure placed on the left of two others, or of two ciphers, expresses a number a hundred times as great as if it were alone.

12. From *nine hundred and ninety-nine*, we can, by the same artifice, count as far as *nine thousand nine hundred and ninety-nine*, in forming of ten hundreds a unit called *thousand*, counting these units as before, and representing them by the same figures placed on the left of the hundreds.

Thus, to represent *seven thousand eight hundred and fifty-nine*, we write 7859; to express *seven thousand and nine*, we write 7009; and for *seven thousand*, we write 7000, where we see that a figure on the left of three others, or of three ciphers, expresses a number a thousand times as great as if it were alone.

13. In continuing thus to include ten units of a certain order in a single unit, and to place these new units in ranks more and more advanced towards the left; we are able to express in a uniform manner, and with ten characters only, all the whole numbers imaginable.

* Or, we can represent all possible numbers with the ten following characters, which are called figures: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0. The nine first represent one, two, three, four, five, six, seven, eight, nine; the character 0, is called nought, and does not represent any number.

To effect this, in proceeding from the right hand towards the left, the first figure is considered as representing units; the second, tens; the third, hundreds; the fourth, thousands; the fifth, tens of thousands; the sixth, hundreds of thousands; the seventh, millions, etc. Also, 0 is put in the place of the units, the tens, the hundreds, the thousands, the tens of thousands, the hundreds of thousands, the millions, etc. which are wanting in a number.

It is evident, in effect, that with these ten characters we can represent all the whole numbers possible, since the following numbers, one, two, three, four, five, six, seven, eight, nine, ten, eleven, etc. which, beginning with a unit, go on in-

creasing by a unit to infinity, may be expressed by these ten characters.

14. To read with facility a number expressed by any number of figures at pleasure, we separate them into periods of three figures each, proceeding from the right hand towards the left: we shall give to these periods the following names, beginning at the right hand, units, thousands, millions, billions, quadrillions, quintillions, sextillions, etc. The first figure of each period (beginning always at the right) will have the name of that period; the second, that of tens; and the third, that of hundreds.

Thus, beginning at the left, we read each period as if it stood alone, and we pronounce at the end of each the name of that same period; for example, to read the following number:

quadrillions	trillions	billions	millions	thousands	units
23	456	789	234	565	456

we say twenty-three *quadrillions*, four hundred and fifty-six *trillions*, seven hundred and eighty-nine *billions*, two hundred and thirty-four *millions*, five hundred and sixty-five *thousands*, four hundred and fifty-six *units*.

From what has been said above, knowing that there are three places of figures, that is to say, units, tens, and hundreds under each name, we can easily write down in figures any number which is expressed in words, by first writing the names of the different periods in succession, beginning with the highest name mentioned in the question, and terminating with the units, and then placing such figures under each name as the nature of the question requires; for example, if it be required to write down in figures the following number, fifty billions, thirty-two millions, five hundred and two thousands, six hundred and fifteen, I write the names of the different periods, beginning with the highest name mentioned in the question, thus,

Billions	Millions	Thousands	Units
50	032	502	615

then as the question requires fifty billions, I write 50 under billions, putting a cipher in the place of units, and leaving the place of hundreds vacant, because a cipher placed on the left

of the 5 would not at all alter its value; then as the question requires thirty-two millions, I place 32 under millions, but I must not here leave the place of hundreds vacant as I did under billions, for as the value of the 5 which stands under billions depends upon the number of places to which it is removed towards the left, that omission would decrease its value ten-fold, I therefore write 032 under millions; again, as the question requires five hundred and two thousands, I write 502 under thousands; and lastly, for the six hundred and fifteen, I write 615 under units, which completes the proposed number.

Let the following numbers be expressed in figures.

1. Four thousand six hundred and twenty.

2. Fifty-six millions three hundred thousand and ten.

3. Seventeen billions, forty-five millions, three thousand and four.

4. One hundred and six trillions, sixty-seven billions, twenty-four thousand and sixty.

5. Three hundred quadrillions, six trillions, seventy billions, three hundred millions and sixteen.

15. From the numeration which we have explained, and which is purely of agreement, it follows that as we advance from the right hand towards the left, the units of which each figure is composed are ten times as great as they would be if the figure stood one place farther to the right; and that, consequently, to render a number ten times, a hundred times, a thousand times, etc. greater, we have only to put one, two, three, etc. ciphers on the right of its unit figure: on the contrary, as we retrograde from the left hand towards the right, the units in each figure are only one-tenth of what they would be if the figure stood one place farther towards the left.

16. Such is numeration: it is the basis of every other method of counting, although in many arts we do not always count by tens, by tens of tens, etc.

17. To estimate quantities less than the unit which we have chosen, we divide this into other smaller units. The number of them is indifferent in itself, provided that we can measure

the quantities we wish to measure; but what we ought principally to have in view in these divisions, is to render the calculations as commodious as possible; for this reason, instead of dividing the unit at first into a great number of parts, in order to estimate the smallest, we first divide it only into a certain number of parts, and then subdivide these into others, and these again into smaller. Thus in money we divide the pound into 20 parts, which we call *shillings*, the shilling into 12 parts, which we call *pence*, and the penny again into 4 parts, which we call *farthings*. In the same manner we divide the bushel into 4 *pecks*, the peck into 8 *quarts*, and the quart into 2 *pints*, so that in the first case we count by fours, by twelves, and by twenties, and in the second by twos, by eights, and by fours.

18. A number which is thus composed of parts related to different units, is called a compound number; and on the contrary, that which contains only one kind of unit, is called a simple number. 8*l.* or 8 pounds is a simple number. 8*l.* 17*s.* 8*d.*, or 8 pounds, 17 shillings, and 8 pence, is a compound number.

19. Every art divides in its own way the principal unit which is chosen. The subdivisions of the acre are different from those of the year, and so on: we will explain these in treating of compound numbers.

20. But of all the divisions and subdivisions which we can make of the unit, that which is made by decimals, that is to say, in dividing the unit into ten parts, and each of these into ten parts, etc., is incontestably the most commodious in calculations. Decimals are much used in the practice of the mathematics, and their formation and calculation are absolutely the same as in ordinary or whole numbers, as will be explained.

21. To estimate in decimals the parts less than the unit, we conceive that this unit, whatever it may be, pound, foot, etc. is composed of ten parts, as we imagine the shilling composed of 12 pence. These new units, in opposition to *tens*, are called *tenths*; we represent them by the same figures as the simple units, and as they are ten times smaller than these, we place them on the right of the simple units.

But to prevent mistakes, and not to give occasion of taking the tenths for the simple units, it was agreed to fix the place of the units, once for all, by a particular mark: that which is most in use, is a comma placed on the right of the figure which represents the units, or, which is the same thing, between the units and the tenths; thus to express *twenty-four units* and *three tenths*, we write 24,3.

22. We may at the same time actually consider the *tenths* as units which have been formed of ten others, each ten times smaller than the *tenths*, and, for the same reason of analogy, place them on the right of the tenths. These new units, ten times smaller than the *tenths*, will be a hundred times smaller than the principal units, and for this reason we shall call them *hundredths*.

Thus, to express *twenty-four units*, *three tenths*, and *five hundredths*, we shall write 24,35.

23. Let us also conceive that the *hundredths* are each formed of ten parts; these parts will be a thousand times smaller than the simple or principal units, and for this reason we shall call them *thousandths*; and as they will be ten times smaller than the *hundredths*, we shall place them on the right of these. In continuing thus to subdivide by ten, we shall form new units, which we shall call successively *ten thousandths*, *hundred thousandths*, *millionths*, *ten millionths*, *hundred millionths*, *billionths*, etc., and which we shall place in ranks more and more advanced on the right of the comma.

24. The parts of a unit which we have here described are those which are called decimals.

25. As to the manner of reading them, it is the same as for whole numbers. After having read the figures which are on the left of the comma, we read the decimals in the same manner; but at the end, we add the name of the decimal units of the last kind: thus, to read the number 34,572, we say thirty-four units and five hundred and seventy-two *thousandths*; if they were feet, for instance, we should say thirty-four feet and five hundred and seventy-two *thousandths* of a foot.

The reason of this is easy to perceive, if we observe that in the number 34,572, the figure 5 may be considered, either

as five *tenths*, or as five hundred *thousandths*, since the tenth (21) is equal to ten *hundredths*, and the hundredth (22) is equal to ten *thousandths*, the tenth will contain ten times ten *thousandths*, or a hundred *thousandths;* therefore, the five *tenths* are equal to five hundred *thousandths*. For the same reason the figure 7 may be read seventy thousandths, since (23) each *hundredth* is worth ten *thousandths*.

26. With regard to the kind of units expressed by the last decimal figure, it will always be easily found in counting successively, from the left hand to the right, upon each figure from the comma the following names: *tenths*, *hundredths*, *thousandths*, *ten thousandths*, etc.

27. If there were no whole units, but only parts of a unit, we should put a cipher to hold the place of the units; thus, to express one hundred and twenty-five *thousandths*, we should write 0,125. If we would express twenty-five *thousandths*, we should write 0,025, putting a cipher between the comma and the other figures, as well to show that there are no *tenths* as to give the other parts their true value. For the same reason, to express six *ten-thousandths*, we write 0,0006.

28. Let us now examine the changes which can be made in a number by displacing the comma.

Since the comma determines the place of the units, and since all the other figures have their value depending upon their distance from this same comma, if we advance the comma one, two, three, etc. places towards the left, we render the number 10, 100, 1000, etc. times less; and on the contrary, we render it 10, 100, 1000, etc. times greater, if we remove the comma one, two, three, etc. places towards the right.

In effect, if we have 4327,5264, and if in advancing the comma one place towards the left, we write 432,75264, it is visible that the thousands of the first number are hundreds in the new; the hundreds are tens; the tens, units; the units, tenths; the tenths, hundredths, and so on. Each figure, therefore, has become ten times less by this remove. If, on the contrary, in removing the comma one place farther to the right, we had written 43275,264, the thousands of the first number would have been changed to tens of thousands, the hundreds to thousands, the tens to hundreds, the units to tens,

the tenths to units, and so on. This new number is then ten times greater than the first.

29. A similar reasoning shows that in advancing towards the left two or three places, we should have rendered the number a hundred or a thousand times less; and on the contrary, a hundred or a thousand times greater, in removing the comma two or three places towards the right.

30. The last observation which we shall here make upon decimals, is, that we do not alter their value in putting on the right of the last decimal figure as many ciphers as we please. Thus, 43,25 is the same as 43,250, 43,2500, or 43,25000, etc.

For as each *hundredth* is equal to ten *thousandths* or a hundred *ten-thousandths*, etc., the twenty-five *hundreths* are equal to two hundred and fifty *thousandths* or two thousand five hundred *ten-thousandths*, etc. In one word, it is the same thing as when instead of saying 25 dollars, we say 250 dimes, 2500 cents, or 25000 mills. See table of Federal Money.

OF THE OPERATIONS OF ARITHMETIC.

31. To add, to subtract, to multiply, and to divide, are the four fundamental operations of arithmetic. All questions relating to numbers require the practice of some or all of these operations. It is important, therefore, to render them familiar, and perfectly to understand their nature.

32. The intention of arithmetic is, as we have already said, to furnish means for the easy calculation of numbers. These means consist in reducing the most complex numbers to the most simple, or those expressed by the least possible number of figures.

OF THE ADDITION OF WHOLE NUMBERS AND DECIMAL PARTS.

33. To express the total value of several numbers by a single number, is what is called *performing an addition*.

When each of the numbers which we propose to add consists of only one figure, we have no need of a rule; but when there are many figures, we find their total value which we call their sum, by observing the following rule.

Write the numbers under each other, so that the unit figures may be under each other in one vertical column; let

the tens be under tens, the hundreds under hundreds, etc. Draw a line underneath.

First, add all the numbers which are in the column of units: if the sum do not exceed nine, write it under that column; if it exceed nine, it will contain ten or a number of tens; write underneath only the excess above the number of tens it contains: consider these tens as so many units, and add them to the next column or column of tens: observe with regard to the sum of the numbers in this second column, the same rule as for the sum of the first, and continue thus from column to column to the last, under which write the whole amount of that column. Let us explain this rule by examples.

EXAMPLE I.

Required the sum of the numbers 54925 and 2023: I write the numbers as here seen - - - 54925

2023

56948 sum.

And having drawn a line underneath, I begin with the units, and say 5 and 3 are 8, which I write under the same column.

I then proceed to the column of tens, saying 2 and 2 are 4, which I write underneath.

In the column of hundreds, I say 9 and 0 are 9, which I write underneath.

In the column of thousands, I say 4 and 2 are 6, which I write under that column.

Lastly, in the column of tens of thousands, I say 5 and 0 are 5, which I write likewise underneath.

The number 56948, found by this operation, is the sum of the two proposed numbers, since it contains all the units, the tens, the hundreds, the thousands, and the tens of thousands, which we have assembled successively.

EXAMPLE II.

Required the sum of the four following numbers: 6903... 7854...953 and 7327; I write the numbers as is here seen:

```
 6903
 7854
  953
 7327
 -----
23037 sum.
```

And commencing as before at the top of the right hand column, I say, 3 and 4 are 7, and 3 are 10, and 7 are 17; I write the 7 units under the first column, and the ten I retain to add as a unit to the numbers in the next column, the units of which are also tens.

Proceeding to the second column I say 1 which I carry and 0 are 1, and 5 are 6, and 5 are 11, and 2 are 13; I write 3 under the same column, and I carry for the ten a unit to the next column, saying: 1 and 9 are 10, and 8 are 18, and 9 are 27, and 3 are 30; I place 0 under that column, and I carry for the three tens, three units which I add to the next column, saying: 3 and 6 are 9, and 7 are 16, and 7 are 23; I write 3 under that column, and as there are no more columns, I advance one place farther to the left the two tens which would belong to the next column if there were another.

34. If there be decimal parts, since these are counted like the other numbers, by tens, as we advance from the right hand towards the left, the rule for adding them is absolutely the same, observing always to place the units of the same order under each other in the same column, in which case the commas or decimal points will also be under each other.

Thus, if we propose to add the three numbers 72,957... 12,8...124,03, I shall write - - - 72,957

```
   12,8
  124,03
  ------
  209,787
```

And in following the above rule, I shall have 209,787 for the sum.

EXAMPLES.

1. Required the sum of twenty-nine; six hundred and fifty-eight; two hundred and thirty-four; and seven thousand seven hundred and eighty-five.

Answer. The sum is eight thousand, seven hundred and six.

2. Required the sum of two and nine tenths; sixty-five, and eight tenths; twenty-three, and four tenths; seven hundred and seventy-eight, and five tenths.

Answer. Eight hundred and seventy, and six tenths.

3. Required the sum of the four numbers six, and nine hundred and three thousandths; seven, and eight hundred and fifty-four thousandths; nine hundred and fifty-three thousandths; seven, and three hundred and twenty-seven thousandths.

Answer. Twenty-three, and thirty-seven thousandths.

4. Required the sum of the three numbers, seven, and two thousand nine hundred and fifty-seven ten-thousandths; one, and twenty-eight hundredths; twelve, and four hundred and three thousandths.

Answer. Twenty, and nine thousand seven hundred and eighty-seven ten-thousandths.

5. Required the sum of the three following numbers, three hundred, and three ten-thousandths; seventy, and eighty-four thousandths; nine, and three tenths.

Answer. Three hundred and seventy-nine, and three thousand eight hundred and forty-three ten-thousandths.

6. Required the sum of the answers to the five preceding examples.

Answer. Ten thousand.

As there are certain signs used by mathematicians to denote the four fundamental operations of arithmetic, we shall consider each of these together with the rules to which it applies.

There is also an auxiliary sign, viz. two equal and horizontal straight lines thus =, which signifies that the numbers be-

tween which it is placed are equal to one another. For example 4=4 is read four equal to four.

A straight cross +, which is called plus, or more, is the sign of addition, and signifies that the numbers between which it is placed are to be added together. Thus 4+3 signifies that 4 is to be added to 3, and is read four plus three, also 4+3=7, is read four plus three equal to seven, and the expression 9+5+8=22, is read nine plus five plus eight equal to twenty-two, in adding the numbers 9, 5, and 8 together.

Note.—It is immaterial in what order they are added as 9+5+8 is the same as 9+8+5 or 8+5+9, etc.

EXAMPLES.

1. 19+9+9+9+9+9+9+9+9+9=100.

It will be seen in this example, that (because 9+1=10) in adding 9 to any significant figure in a number, that figure will be diminished and the next figure increased, each by a unit: Thus 19+9=28; 28+9=37; 37+9=46; 46+9=55; 55+9=64, etc. where in each new addition the unit figure is one less and the figure in the place of tens one greater.

2. 27+36+107+9=179.

3. 93,7+54,36+0,007=148,067

When there are only two or three numbers, the addition may easily be performed without placing the numbers under each other, by commencing with the lowest units and being careful to add only those of the same order to each other, carrying for the tens as usual.

4. 75+62,3+58,9=196,2.

5. 556+829,43+66,078+3,008+5,833=1460,349.

6. 725+596+274+592+673+889+438+299=4486.

In order to prove the work, when the addition is somewhat long, as in the present example, it is usual to add both ways; that is, after having placed the numbers, to add from the top downward and from the bottom upward, if the sum be the same both ways, the work can scarcely be wrong.

ANOTHER METHOD OF PROVING ADDITION.

This proof of addition is made in adding anew, by parts, but in beginning at the left, the sums which we have already added. We take the sum of the first column, from that part of the whole sum which answers to it: we write the remainder underneath, which we consider as so many tens, to join it to the following figure of that sum, and from the whole we take the sum of the column above it, we continue thus to the column of units, the sum of which being subtracted there should be no remainder.

Thus, having found that the four numbers - - 6903
7854
953
7327

give the sum 23037
....
3110

To verify this result, I add the same numbers in commencing at the left, and I say, 6 and 7 are 13, and 7 are 20, this taken from 23, leaves 3 or three tens, which with the cipher following makes 30. I pass to the next column, and I say, 9 and 8 are 17, and 9 are 26, and 3 are 29, which I take from 30; there remains 1 or ten, which being added to the next figure makes 13. I add all the figures in the third column saying, 5 and 5 are 10, and 2 are 12, this taken from 13, leaves 1 or ten, which added to the figure 7, makes 17. I add likewise all the figures in the column of units, saying, 3 and 4 are 7, and 3 are 10, and 7 are 17, which taken from 17 leaves nothing; whence I conclude that the first operation is correct.

We have reason to conclude that the first operation was correctly performed, since after this proof there remains nothing, because having taken away successively all the thousands, all the hundreds, all the tens, and all the units, of which we had composed the sum, there should nothing remain.

7. 7+8+6+5+9+8+7+8+2+4+6+8+9+7+3+6+9+8+2+6=128.

In adding any question, whenever two figures succeed each other, the sum of which is ten, it will facilitate the operation to comprehend them both together; thus, in the present example I say, 7 and 8 are 15, and 6 are 21, and 5 are 26, and 9 are 35, and 8 are 43, and 7 are 50, and 10 (comprehending the 8+2) are 60, and 10 (comprehending 4+6) are 70, and 8 are 78, and 9 are 87, and 10 (comprehending 7+3) are 97, and 6 are 103, and 9 are 112, and 10 (comprehending 8+2) are 122, and 6 are 128.

OF THE SUBTRACTION OF WHOLE NUMBERS AND DECIMAL PARTS.

35. Subtraction is the operation by which we take one number from another. The result of this operation is called *remainder*, *excess*, or *difference*.

To perform this operation, we write the number which we would take away under the other, in the same manner as in addition; and having drawn a line underneath the whole, we take away, in going from the right hand towards the left, each lower figure from its correspondent upper one; that is to say, the units from units, the tens from tens, etc.; we write each remainder underneath, and a cipher when nothing remains.

When the lower figure is greater than the corresponding upper one, we add ten units to the upper one in borrowing by the thought a unit from the next figure on the left, which should be regarded as one unit less in the next operation.

Instead of diminishing by a unit the figure from which we have borrowed, we can, if we please, leave it as it is, and increase by a unit that which is to be taken from it: the remainder will always be the same.

EXAMPLE I.

We propose to take 5432 from 8954. I write these two numbers as follows: - - - - - - - 8954

```
 8954
 5432
 ----
 3522
```

And in commencing with the unit figure, I say 2 from 4 there remains 2, which I write underneath; again passing to the tens, I say 3 from 5 there remains two, which I write under the tens. At the third column, I say 4 from 9 leaves 5, which I write under that column. Lastly, at the fourth, I say 5 from 8 leaves 3, which I write under 5, and I have 3522 for the remainder of 5432 taken from 8954.

EXAMPLE II.

If we would take 7987 from 27646,

we write - - - 27646
7987
———
19659

As we cannot take 7 from 6, we add ten units to the 6 which we borrow in taking a unit from the figure 4, and we say 7 from 16 leaves 9, which we write under 7.

Passing to the tens, we say not 8 from 4, but 8 from 3 only, because in borrowing we have diminished 4 by a unit: as we cannot take 8 from 3, we add as before ten units to the 3 which we borrow in taking a unit from the figure 6 on the left, and we say 8 from 13 leaves 5, which we write under 8.

Passing to the third column, we say in the same way, 9 from 5, or rather 9 from 15, (in borrowing as before) leaves 6, which we write under 9.

At the fourth column, we say for the same reason, 7 from 6, or rather 7 from 16, leaves 9, which we write under 7; and as there is nothing to take from the fifth column, we write under this column not 2, because we have borrowed a unit from this 2, but only 1, and we have 19659 for the remainder.

36. If the figure from which we should borrow be a cipher, we must borrow not from this cipher, but from the first significant figure on the left of it; now although this be to borrow 100, 1000, or 10000, according as there may be one, two, or three ciphers following each other, we operate notwithstanding as before; that is to say, we add only ten to the figure for which we have borrowed; and as these ten are considered as taken from the 100, or 1000, etc. which we have borrowed, to employ the 90, or 990, etc. which remain, we

consider the following ciphers as so many nines; this the following example will illustrate.

EXAMPLE III.

	• •
	99
If from - - -	20064
we would take - -	17489
	2575

We first say, 9 from 4, or rather from 14, (in borrowing from the next figure) leaves 5. Then, to take 8 from 5, as that cannot be done, and as it is not possible to borrow from the next figure which is a cipher, we borrow a unit from the two, which is worth one thousand with regard to the figure upon which we are operating. Of this thousand we take ten only which we add to 5, and we say 8 from 15 leaves 7.

As we have employed only ten units of the thousand which we borrowed, we employ the remaining 990 in subtracting from it the figures which stand under the ciphers; this is the same as to consider each cipher as if it were 9. Thus we say, 4 from 9 leaves 5: then 7 from 9 leaves 2: and lastly, 1 from 1 leaves nothing.

When the figure from which we would subtract, is less than the lower figure, we may add ten to the upper figure without borrowing from the next figure on the left, in which case we must add a unit to the next figure on the left in the lower number; the numbers being thus both increased by the same number, their difference will remain the same.

EXAMPLE IV.

If it be required to take 7583 from 9132,

I write - - -	9132
	7583
	1549

After which, because 3 cannot be taken from 2, I add ten to 2 which makes 12: I then say 3 from 12 leaves 9, which I write under 3: and proceeding to the next column, I add 1

to 8 which makes 9, and as 9 is greater than 3, adding ten as before, I say 9 from 13 leaves 4, which I write underneath. At the third column, I add 1 to 5 which makes 6, then as 6 is greater than 1, I add ten, and say 6 from 11 leaves 5, which I write underneath. Lastly, at the fourth column, I add 1 to 7 which makes 8, and 8 taken from 9 leaves 1; I have therefore 1549 for the remainder.

It will be easily seen that by this method the numbers are both increased alike, because in the above example the 1 which we add to the figure 8 in the lower number being in the place of tens is equal to the ten units which we added to the figure 2 in the upper number; the 1 which we add to the figure 5 in the lower number being in the place of hundreds is equal to the ten tens which we added to the figure 3 in the upper number; lastly, the 1 which we add to the 7 being one thousand, is equal to the ten hundreds which we added to the figure 1 in the upper number.

It is also evident, since the numbers added to each of the two proposed numbers are equal, that in subtracting the one of these equal numbers from the other, there will nothing remain, and consequently, this method of operating has no effect upon the difference primarily existing between the two proposed numbers.

37. If there be decimal parts in the numbers upon which we would operate, we shall follow the very same rule; but to avoid embarrassment in the application of this rule, we have only to render the number of decimal figures the same in each of the two proposed numbers, in placing a sufficient number of ciphers on the right of that which has the fewest decimals: this preparation will not at all alter the value of the number (30.)

EXAMPLE V.

From - - -	5403,25
we would take - -	385,6532

I place two ciphers on the right of the decimals in the upper number; after which I operate upon the two numbers thus prepared, precisely according to the method given in the preceding example:

```
 5403,2500
  385,6532
 ---------
 5017,5968
```

And I find for the remainder 5017,5968.

EXAMPLES.

1. What is the difference between ten thousand five hundred, and nine thousand one hundred and ninety-seven?

Answer. One thousand three hundred and three.

2. What is the difference between thirty-nine and five hundredths, and twenty-seven and three hundred and twenty-five thousandths?

Answer. Eleven and seven hundred and twenty-five thousandths.

3. What is the difference between eighty, and seventy-nine and six hundred and fifty-four thousandths?

Answer. Three hundred and forty-six thousandths.

4. What is the difference between the sum of the five numbers under examples for practice in numeration (page 9) and a quintillion?

Answer. Six hundred and ninety-nine quadrillions, eight hundred and eighty-seven trillions, eight hundred and forty-five billions, five hundred and ninety-eight millions, six hundred and sixty-eight thousands, two hundred and ninety.

A short horizontal line thus —, which is called minus, or less, is the sign of subtraction, and signifies that the latter of the two numbers between which it is placed is to be taken from the former: thus 7—4 shows that the number 4 is to be taken from 7, and is read seven minus four; also 7—4=3 is read seven minus four equal to three.

EXAMPLES.

1. 14—8=6.	4. 100,32—67,505=32,815.
2. 63—37=26.	5. 350,6—50,76=299,84.
3. 27—26,99=0,01.	6. 100—0,6053=99,3947.

The sign plus, is also called by mathematicians the *positive* sign, and the sign minus the *negative* sign; accordingly, a number having the sign + plus prefixed to it is called a

positive number, and a number having the sign — minus prefixed to it is called a *negative* number.

Note. All numbers having no sign prefixed to them are positive.

These two signs, plus + and minus —, that is to say, the two operations of addition and subtraction, destroy the effect of each other. Thus $5+3-3=5$; where it is seen that having taken away the same that we added, the number 5 is still the same.

The expression $5+3-3$ is the same as $5-3+3$ or $3-3+5$, etc. for in the first, $5+3=8$ and $8-3=5$; in the second, $5-3=2$ and $2+3=5$; and in the third $3-3=0$ and $0+5=5$, where it is seen that it is immaterial in what order the numbers stand, provided that they still retain the same sign.

When there are several positive and several negative numbers in the same expression, add all the positive numbers together and all the negative numbers together, the difference between these two sums, preceded by the sign of the greater, will be the value of the expression.

Thus in the following expression $5+4-6+9-3$.

$5+4+9=18$ the sum of all the positive numbers.

$6+3=9$ the sum of all the negative numbers.

Then $18-9=9$ the value of the expression $5+4-6+9-3$.

Or thus $5+4-6+9-3=5+4+9-6-3=18-9=9$.

EXAMPLES.

1. $6+7+8-3=21-3=18$.
2. $9-5+2-3+6=9+2+6-5-3=17-8=9$.
3. $961+39-0,73=999,27$.
4. $22,5+1,235-25,376-74,624+76,265=$

OF MULTIPLICATION.

40. To multiply a number by another is to take the first of these two numbers as many times as the other contains a unit. To multiply 4 by 3 is to take three times the number 4.

41. The number which we multiply is called the *multiplicand;* that by which we multiply is called the *multiplier;* and the number found by the operation is called the *product.*

42. The word *product* has commonly a much more extensive signification, but we shall use it only to designate the result of a multiplication.

The multiplicand and multiplier are also called the factors of the product: thus, 3 and 4 are the factors of 12, because 3 times 4 are 12.

43. Pursuing the idea which we have given of multiplication, we see that we could perform this operation by writing the multiplicand as many times as there are units in the multiplier, and afterward performing the addition. For instance, to multiply 7 by 3, we could write $7+7+7$, and the sum 21, resulting from this addition, would be the product.

But when the multiplier is somewhat considerable, the operation becomes very long. That which we properly call multiplication is the method of arriving at the same result in a shorter way.

44. When the two proposed numbers are only considered in an abstract manner, that is to say, without regard to the nature of their units, it matters little which of them we take for multiplicand or for multiplier; for example, if we have 4 to multiply by 3, it is indifferent whether we multiply 4 by 3, or 3 by 4; the product will always be 12.

In effect, 3 times 4 is nothing else than the triple of once 4, and 4 times 3 is the triple of 4 times 1. It is evident that once 4 and 4 times 1 are the same thing; and we might apply the same reasoning to every other number.

45. But when, by the declaration of the question, the *multiplier* and the *multiplicand* are concrete numbers, it is important to distinguish the multiplicand from the multiplier; this attention is most necessary in the multiplication of compound numbers, of which we shall speak hereafter.

The question which leads to the multiplication always makes known which is the quantity that should be repeated many times, that is to say, the multiplicand; and which the quantity that shows how many times we should repeat the multiplicand, that is to say, the multiplier.

46. As the multiplier is destined to show how many times we should take the multiplicand, it is always an abstract number: thus, when we demand how much 25 cords of wood

should cost, at the rate of 15 dollars per cord, we see that the multiplicand is 15 dollars, which must be repeated 25 times, whether 25 signifies cords of wood or any thing else.

47. The product which is formed of the repeated addition of the multiplicand, will then have its units of the same nature as those of the multiplicand.

After this little digression on the nature of the units of the product and its factors, let us return to the method for finding this product.

48. The rules for the multiplication of the most complicate numbers, are reduced to the multiplying of a number expressed by a single figure by a number expressed by a single figure. One must therefore exercise himself to find the product of the numbers expressed by a single figure, in adding successively a number to itself. We can also, if we choose, make use of the following table.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

The first line of this table is formed by adding 1 to itself ssively.

The second in adding 2, in the same manner.

The third in adding 3, and so on for the rest.

49. To find, by means of this table, the product of two numbers, we will seek one of these numbers, the multiplicand for example, in the upper line; and, leaving this number, we will descend vertically till we come opposite the multiplier, which we shall find in the first column: the number upon which we rest will be the product. Thus, to find, for example, the product of 9 into 6, or how much is 6 times 9? I descend from 9, taken in the first line, till opposite 6, in the first column; the number upon which I rest is 54: consequently, 6 times 9 are 54.

This is enough to go on with the multiplication of numbers expressed by many figures.

OF MULTIPLICATION BY A NUMBER OF ONE SINGLE FIGURE.

50. Write the multiplier, which we suppose here to be a single figure, under the multiplicand, not much matter under what figure; but to fix the ideas, let us suppose that it be under the figure of units.

First, multiply the number of the units by the multiplier; and if the product only contain units, write this product underneath; if it contain units and tens, write only the units, and, counting the tens as so many units, retain these.

Multiply, in the same manner, the number of the tens in the multiplicand, and to the product add the units which were retained; write the whole underneath, if it can be signified by a single figure; if not, write only the units of this product, and retain the tens of it, which are hundreds, to add them as units to the next product, which will also be hundreds.

Continue to multiply successively, following the same rule, all the figures of the multiplicand; the series of figures which you will have written will be the product.

EXAMPLE I.

We demand how many feet there are in 2864 fathoms. Now, as a fathom is 6 feet, the question requires that we should take 6 feet 2864 times; or, which is the same, (44) 2864 feet six times.

I write then - - 2864
6

7184 Product.

And I say, in commencing with the units, 6 times 4 are 24; I write 4, and I retain 2 units for the 2 tens.

2dly. 6 times 6 are 36, and 2 I retain are 38; I set down 8, and retain 3.

3dly. 6 times 8 are 48, and 3 I retain are 51; I write 1. and retain 5.

4thly. 6 times 2 are 12, and 5 I retain are 17; which I write altogether; because there is nothing more to multiply. The number 17184 is the required product, or the number of feet which are equal to 2864 fathoms, since it contains 6 times the 4 units, 6 times the 6 tens, 6 times the 8 hundreds, and 6 times the 2 thousands, and, consequently, 6 times the number 2864.

OF MULTIPLICATION BY A NUMBER CONSISTING OF MANY FIGURES.

51. When the multiplier contains many figures, we must do with each of these successively what we have prescribed when there is but one, but in commencing always at the right hand. Thus, we shall first multiply all the figures of the multiplicand by the unit figure of the multiplier, then by that of the tens, and we shall not write the second product directly under the first; but as it ought to be a number of tens, since it is by the tens that we multiply, we shall place the first figure of this product under the tens; and the other figures, always advancing towards the left.

The third product, which is found in multiplying by the hundreds, will be placed under the second, but in advancing it one place towards the left. We will follow the same law for the others.

All the multiplications being made, we add the several products which they have given, and the sum is the total product.

EXAMPLE II.

We propose to multiply 65487
by - - - - 6958

523896	eight times	65487.
327435	fifty times	——.
589383	nine hundred times	——.
392922	six thousand times	——.
455658546	Product.	

I first multiply 65487 by the number 8 of the units of the multiplier, and I write successively under the line the figures of the product 523896 which I find in following the rule given for the first case (50.)

I multiply in the same way 65487 by the second figure 5 of the multiplier, and I write the product 327435 under the first product, but in placing the first figure 5 under the tens of this first product.

Multiplying likewise 65487 by the third figure 9, I write the product 589383 under the preceding, but in placing the first figure 3 in the place of hundreds, because the number by which I multiply is a number of hundreds.

Lastly, I multiply 65487 by the last figure 6 of the multiplier, and I write the product 392922 under the preceding, in advancing it yet one place farther towards the left, so that the first figure may occupy the place of thousands, because the figure by which I multiply signifies thousands. I then add all these products, and I have 455658546 for the product of 65487 multiplied by 6958, that is to say, for the value of 65487 taken 6958 times.

In effect, we have taken 65487, 8 times by the first operation, 50 times by the second, 900 times by the third, and 6000 times by the fourth.

52. If the multiplicand or the multiplier, or both, be terminated by ciphers, we abridge the operation in multiplying as if the ciphers were not there; but we afterward place them all on the right of the product.

EXAMPLE III.

We propose to multiply 6500
by - - - - 350

325
195

2275000

I multiply only 65 by 35, and I find 2275, on the right of which I write the ciphers which are found in all, on the right of the multiplicand and multiplier.

In effect, the multiplicand 6500 represents 65 hundreds: therefore when we multiply 65, we should understand that the product is hundreds. Likewise, the multiplier signifies 35 tens. Therefore when we multiply by 35, we should understand that the product will be tens; it will then be tens of hundreds, that is to say, thousands; it ought therefore to have three ciphers on the right. We will apply a similar reasoning to all other cases.

53. When ciphers are found between the figures of the multiplier, as the multiplication by these ciphers could only give ciphers, we will not write these in the product; but going directly to the first significant figure on the left of these ciphers, we shall advance the product towards the left as many places, plus 1, as there are ciphers on the right of it in the multiplier, that is to say, two places if there be one cipher, three if there be two, etc.

EXAMPLE IV.

If we have - - - 42052
to multiply by - - - 3006

252312
126156

126408312

After having multiplied by 6, and written the product 252312, we multiply directly by 3; but we write the pro-

duct 126156, in such manner that it shall signify thousands; we therefore advance it three places, that is to say, one place farther than there are ciphers interposed between the figures of the multiplier.

OF THE MULTIPLICATION OF DECIMAL PARTS.

54. To multiply decimal parts, we observe the same rule as for whole numbers, without any regard to the comma; but after having found the product, we separate upon the right hand by a comma, as many figures as there are decimals in both the multiplicand and multiplier.

EXAMPLE I.

We propose to multiply - -	54,23
by - - - - - -	8,3
	16269
	43384
	450109

I multiply 5423 by 83, the product is 450109; and as there are two decimals in the multiplicand, and one in the multiplier, I separate three figures on the right of the product, which becomes 450,109, what it ought to be.

The reason of this rule is easy to discover, in observing that if the multiplier were 83, the decimals of the product would only contain *hundredths*, since we should have repeated the multiplicand 54,23, the decimals of which are hundredths, 83 times; but as the multiplier is 8,3, that is to say (21) ten times less than 83, the product should contain units ten times less than hundredths; the last figure then of these decimals should be (23) *thousandths*, there should then be three decimal figures in this product, that is to say, as many as there are in both the multiplicand and multiplier.

We can apply a similar reasoning to every other case.

EXAMPLE II.

If we have - - - -	0,12
to multiply by - -	0,3
	0,036

We multiply 12 by 3, which gives 36.

As the rule prescribes the separation of three figures, we might be embarrassed to perform it, since in this product 36 there are but two; but if we again take up the reasoning applied to the preceding example, we shall easily see that there must be, as above, a cipher placed between 36 and the comma. In effect, if we had 0,12 to multiply by 3, it is evident that we should have 0,36; but as we have to multiply only by 0,3, that is to say by a number ten times less than three, we should have a product ten times less than 0,36, that is to say, thousandths, which requires (28) that we should write 0,036.

55. Decimals being generally used as an easy means of finding a sufficient approximation which we substitute for the true result, it will not be unprofitable here to show a method of abridging the operation when we only require the product of two numbers to a proper degree of exactness.

Let us suppose, for example, that having 45,625957 to multiply by 8,6352, I have need only of having the product correct to within a thousandth. I write these two numbers as we see below, that is to say, after having reversed the order of the figures of one of the two, I write it under the other, so that its unit figure may be under the decimal immediately inferior by two degrees to that to which I would limit my product. I then perform the multiplication, neglecting in the multiplicand all the figures which are found on the right of that by which I multiply; and as I change the figure of the multiplier, I always carry the first figure of the new product under the first figure of the preceding. The addition of all these products being performed, I suppress the two last figures, observing however to increase the last of those which remain, by a unit, if the two which I suppress exceed 50;

after which I place the comma to the rank fixed by the species to which I proposed to limit the product.

EXAMPLE III.

I would multiply - - 45,625957
by - - - - - - 8,6352
but I only require to have the product correct to within a thousandth of a unit.

I write the numbers thus:

```
45,625957
   25368
 --------
36500760 Product by 8.
 2737554 — by 6 tenths.
  136875 — by 3 hundredths.
   22810 — by 5 thousandths.
     912 — by 2 ten thousandths
 --------
393,98911
```

Pr. within a thousandth 393,989.

If we had performed the multiplication as usual, we should have had 393,9892638864, which agrees with the preceding as far as the third decimal, as was proposed.

If there were not enough of decimal figures in the multiplicand to make the unit figure of the multiplier correspond to the figure prescribed by the rule, we should supply the defect with ciphers.

Example IV.

We would multiply - 54,236
by - - - - - 532,27
and we would have the product correct to a hundredth.

I write - - - 54,236000
72235

271180000
16270800
1084720
108472
37961

288681953

Product 28868,20 in adding a unit the last figure, because the two which we have suppress exceed 50.

Example V.

Suppose we had to multiply 0,227538917 by 0,566417 and that we would only have 7 decimals in the product.

We shall write - 0,227538917
87146650

.....0
113769455
13652334
1365228
91012
2275
1589
176

128882069

Product 0,1288821

A number that is a certain number of times another number, is called a multiple of that other number. Thus 12 is

multiple of 8, because it contains 3 a certain number of times exactly; also a number that is a multiple of several numbers, is called a common multiple of those numbers; thus 12 is a common multiple of 6, 4, 3, and 2.

When one part of the multiplier is a multiple of the other, the work may often be considerably shortened, as will be seen in the following examples.

Example I.

Suppose that we have 34625 to multiply by 248, I write the numbers as usual,

```
      34625
        248
    -------
     277000  eight times          -   -   34625.
     831000  two hundred and forty times ———.
    -------
    8587000
```

and having multiplied by the unit figure 8 of the multiplier, I have 277000 for the first partial product. Then as the part 24 of the multiplier is a multiple of 8, since it contains 8 just 3 times; instead of multiplying by the figures 24 as usual, I multiply the first product 277000, which is 8 times the multiplicand, by 3, and I have 831000 for 24 times the multiplicand; the unit figure of the last product being put in the place of tens, makes the whole of this last product signify 240 times the multiplicand: wherefore the sum of the two products is 248 times the multiplicand, as was required.

Example II.

What is the product of 358976 multiplied by 147497?

```
     358976
     147497
  ---------
    2512832  seven times the multiplicand.
   17589824  seven times seven times, or forty-nine times.
  52769472   three times forty-nine times, or 147 times.
  ---------
52947883072  Product.
```

Having multiplied by the unit figure 7 of the multiplier, I have 2512832 for the product; then as 49 on the left is just 7 times 7, I take 7 times 2512832, which gives me 17589824 for 49 times the multiplicand, this product being removed one place towards the left is 490 times the multiplicand; again, because 147, which is on the left of 49, is just three times 49, I multiply the product 17589824 by 3, and I have 52769472 for 147 times the multiplicand, the unit figure of this product being put in the place of thousands makes the whole signify 147000 times the multiplicand. Now, because $147000+490+7=147497$, the sum of these three products is 147497 times the multiplicand, as was required.

The sign used to denote multiplication, is formed of two straight lines crossing each other obliquely, thus $\times$. This sign is called *into*, and signifies that the numbers between which it is placed are to be multiplied together: for example, 4×3 signifies that 4 is to be multiplied by 3, and is read four *into* three, and $4\times3=12$ is the operation performed, which is read four into three equal to twelve.

If there be several numbers having the sign $\times$ into between them, multiply any two of them, multiply this product by the next number, and this product again by the next, and so on till all the numbers are involved.

Ex. $7\times4\times2\times6=336$, which is found thus, $7\times4=28$, and $28\times2=56$; lastly, $56\times6=336$; or thus, $6\times2=12$; $12\times4=48$, and $48\times7=336$. It is therefore of no consequence as to the result, in what order they are multiplied.

Examples.

1. $33\times6\times4\times7=5544$.
2. $45\times63\times95\times363=97764975$.
3. $83{,}5\times64{,}2\times5680=30448776$.
4. $3700\times150=555000$.
5. $49765\times7008=348753120$.
6. $56423\times96004=5416833692$.
7. $18296\times567=10373832$.
8. $678153\times6482=4395787746$.
9. $3964\times789=3127596$.

10. $69358 \times 653 = 45290774$.

11. $2968763 \times 5842 = 17343513446$.

12. $8896,5874 \times 819 = 7283,05[illegible]$

A vinculum is a bar ———, or parenthesis [illegible] made use of to collect several quantities into one. Thus $\overline{4+2} \times 3$, or $(6+4)\ 3$, signifies that the sum of 6 and 4 is to be multiplied by 3. Also $\overline{6-4} \times 3$, or $(6-4)\ 3$, signifies that the difference between 6 and 4 is to be multiplied by 3.

EXAMPLES FOR THE EXERCISE OF THE PRECEDING RULES AND ALSO OF THE SIGNS.

1. $6+4 \times 3=18$ but $\overline{6+4} \times 3=30$. It may be observed here that 3 times the sum 10 of the numbers 6 and 4 is the same as the sum of the two products when the numbers 6 and 4 are multiplied separately by 3. Thus $6 \times 3=18$ and $4 \times 3=12$ and $18+12=30$. Also if 10 be divided into any other parts, no matter how many, and if each of these be multiplied by 3, the sum of the products will be 30. Thus $(5+2+3)\ 3=15-6-9=30$.

2. $(7-2)\ 3=15$ and $7-2 \times 3=1$.

3. $37,9+1,35 \times 6=46$, and $(9-4-3)\ 2=13-7-4=2$.

4. $(6,253-4,25+3,997)\ 6=36$.

5. $56-(4,025+3,975)\ 6=2$.

6. $156-23,7+64,35-(16-1,25)\ 7=[illegible]$.

A SHORT WAY OF MULTIPLYING BY ELEVEN.

Suppose, for example, that we would multiply 4332 by 11. I write the multiplicand with a line underneath thus

4332

———

47652

and commencing with the unit figure 2, I write it under the line; then going regularly towards the left, I add each figure to the next figure on the right of it, thus 3 and 2 are 5, which I write under 3; then 4 and 3 are 7, which I write under 4; and lastly, I place the last figure 4 of the multiplicand in the [illegible]

duct one place farther to the left. The reason of this will easily be understood as follows:

43520 is ten times the multiplicand
4352 is once

47872 Sum, or 11 times the multiplicand.

Here we see that in placing any number under ten times that number its unit figure stands under a cipher, and that each of the other figures stands under the next figure on the right of it; also that the last figure is projected one place towards the left.

If the sum of any two figures equal or exceed 10 carry 1 to the next figure on the left.

In like manner if we place once any number under ten times that number and subtract, we shall have nine times that number. Wherefore if I have the above number 4352 to multiply by 9. I write the number with a line underneath as above - - - - - - 4352

39168

and beginning with the unit figure I say 2 from 10 leaves 8, I write 8 under 2 and carry 1 to the next figure on the left, then 1 and 5 are 6, and as I cannot take 6 from 2, I say 6 from 12 leaves 6 which I write under 5, and carrying 1 to 3 which makes 4, I say 4 from 5 leaves 1, which I write under 3, then as I cannot take 4 from 3, I say 4 from 13 leaves 9, which I write under 4, I carry 1, and as there are no more figures, I subtract this 1 from the last figure 4, the remainder 3 I place on the left of 9, which completes the operation.

TO MULTIPLY BY A NUMBER CONSISTING OF ANY NUMBER OF NINES.

Place as many ciphers on the right of the given multiplicand, as there are nines to multiply by, from this new number subtract the multiplicand; the remainder will be the required product.

EXAMPLE.

If I have 6932 to multiply by 999

I write - - - 6932000
6932
———
6925068

In placing three ciphers on the right, I multiply by 1000 (Bezout 15,) and consequently, as 1000—1=999, in subtracting once the number from a thousand times, the remainder is 999 times, as was required.

ON SOME OF THE USES OF MULTIPLICATION.

56. We do not propose to make known all the uses of multiplication; we will only indicate some of them which will serve as a guide to the rest.

Multiplication serves to find the total value of many units, when we know the value of each. For example, 1st. how much should 5842 barrels of flour cost, at the rate of 13 dollars per barrel? We must multiply 13 dollars by 5842, or (44) 5842 dollars by 13; we shall have 7594 dollars, for the required cost. 2d. What is the weight of 5954 cubic feet(1) of water, supposing the cubic foot to weigh 72 pounds? We must multiply 72 pounds by 5954, or 5954 pounds by 72: we shall have 428688 pounds for the weight of the 5954 cubic feet.

57. We employ multiplication to convert the units of a certain kind to units of an inferiour kind. For example, to reduce pounds to shillings, these to pence, and these last to farthings; days into hours, these into minutes, and these last into seconds; as we often have need of these conversions, we shall give some examples of them.

If we require the conversion of 8*l.* 17*s.* 7*d.* into pence; as the pound is worth 20*s.*, we multiply the 8*l.* by 20 (52;) which gives 160*s.*, to which adding the 17*s.* we have 177*s.*; these we multiply by 12, because each shilling is worth 12

(1) The cubic foot is a measure of one foot in length, one foot in breadth, and one foot in height, with which we estimate the capacity of bodies.

pence, and we have 2124 pence, which added to the 7 gives 2131 pence, for the value of 8*l.* 17*s.* 7*d.* converted into pence.

If we demand how many minutes are contained in a common year, or 365d. 5 h. 48m.; as the day contains 24 hours, we shall multiply 24h. by 365, and to the product 8760 we will add 5 h.; we will multiply the total 8765h. by 60 (52,) because the hour contains 60 minutes, and we shall have 525900 minutes, to which adding 48 minutes, we shall have 525948 for the number of minutes contained in a common year.

The operation at length would stand in the following form.

```
  d.     h.   m.
 365     5    48
  24
 ----
 1465
 730
 ----
 8765
    60
 ------
 525948
```

I have here added the 5h. in multiplying by 24. I have also added the 48 minutes in multiplying by 60.

As all such questions will easily be performed and understood from what has been said above, to give more of them here would be but a useless repetition.

58. Before we conclude what regards multiplication, let us observe to beginners, that the double, the triple, the quadruple, etc. of a number, is two, three, four, etc. times that number.

OF THE DIVISION OF WHOLE NUMBERS, AND OF DECIMAL PARTS.

59. To divide a number by another is, in general, to seek how many times the first of these two numbers contains the second.

The number which we divide is called *dividend*: that by which we divide, *divisor*; and that which shows how many times the dividend contains the divisor, is called *quotient*.

We have not always in view in division, to know how many times a number contains another; but we perform the operation in every case as if it tended to this end, for this reason we can, always, consider it as the operation by which we find how many times the dividend contains the divisor.

Hence, it follows, that if we multiply the divisor by the quotient, we shall reproduce the dividend, since this is to take the divisor as many times as it is contained in the dividend: this is general, whether the quotient be a whole number, or a fractional number.

As to the species of the units in the quotient, it is neither by the species of those in the dividend, nor of those in the divisor that we must judge; for the dividend and the divisor remaining the same, the quotient, which will always be the same numerically, may be very different in the nature of its units, according to the question which gives rise to the division.

For example, if we would know how many times 8 dollars contain 4 dollars, the quotient will be an abstract number, which will signify 2 times; but if it be required to know how much work may be done for 8 dollars, at the rate of 4 dollars per square yard, the quotient will be 2 square yards, which is a concrete number, and of which the species has no relation to the dividend nor to the divisor.

But we see, at the same time, that the question alone which leads us to make the division, decides the nature of the units of the quotient.

OF THE DIVISION OF A NUMBER CONSISTING OF SEVERAL FIGURES, BY A NUMBER WHICH HAS BUT ONE.

60. The operation which we are about to describe, supposes that we know how to find how many times a number of one or two figures contains a number consisting of a single figure. This is a knowledge already acquired, when we know by memory the products of the numbers which have but one figure. We can also, to come at this, make use of

the table which we have given above (48.) For example, if I would know how many times 74 contains 9, I seek the divisor 9 in the upper line, and I descend vertically till I find the number nearest to and less than 74, this is 72; then the number 8 which is found opposite to 72 in the first column, is the number of times, or quotient sought.

This supposed, we perform the division of a number which has many figures, by a number which has but one, as follows.

Write the divisor by the side of the dividend, separate the one from the other by a line, and draw a line under the divisor, under which you will write the figures of the quotient, as they shall be found.

Take the first figure on the left of the dividend, or the two first figures, if the first does not contain the divisor.

Seek how often the divisor is contained in this first figure, or these two first figures, write the number of times under the divisor.

Multiply the divisor by the quotient which you have written, and carry the product under the part of the dividend, which you have employed.

Lastly, subtract the product from the part of the dividend above it to which it answers, and you will have a remainder.

On the right of this remainder bring down the next figure of the principal dividend, and you will have a second partial dividend, upon which you will operate as upon the first, placing the quotient on the right of that which is already found, multiplying the divisor again by this quotient, and writing and subtracting the product as before.

You will bring down as before to the side of the remainder of this division, the figure of the dividend which follows that which you have already brought down, and you will continue always in the same manner to the last inclusive.

This rule will be illustrated by the following example.

We propose to divide 8769 by 7.

I write these two numbers as follows:

```
Dividend | 7 Divisor
    8769 | 1252 5/7 Quotient.
    7
   ----
    17
    14
   ----
     36
     35
    ----
      19
      14
     ----
       5 Remainder. or part of the dividend
```

which has not been divided.

In commencing at the left of the dividend, I should say, in 8 thousand how many times 7; but I say simply, in 8 how many times 7? it is contained once. This 1 is naturally one thousand, but the figures which come after give it its real value; this is the reason that I write simply 1 under the divisor.

I multiply the divisor 7 by the quotient 1, and I carry the product 7 under the part 8 which I have divided; making the subtraction, I have 1 for the remainder.

This remainder 1 is the part of 8 which has not been divided, and is ten with regard to the next figure 7; for this reason I bring down the figure 7 on the right of it, and I continue the operation, saying, in 17 how many times 7? twice. I write this 2 on the right of the first quotient 1, found by the first operation. I multiply the divisor 7 by the quotient 2 which I have just found, as in the first operation; I carry the product 14 under my partial dividend 17, and having subtracted there remains 3 for the part which has not been divided.

By the side of this 3, I bring down 6, the third figure of the dividend, and I say, in 36 how many times 7? 5 times; I write 5 in the quotient.

I multiply the divisor 7 by 5; and having written the product 35 under my new partial dividend, I subtract and there remains 1.

Again, by the side of this remainder 1, I bring down the

figure 9 of the dividend, and I say, in 19 how many times 7? twice; I write 2 in the quotient.

I multiply the divisor 7 by this new quotient 2, and having written the product 14 under my last partial dividend 19, I subtract and there remains 5.

I find then that 8769 contains 7 as many times as the quotient that we have written signifies, that is to say, 1252 times, and that there remains 5.

With regard to this remainder, we will be satisfied for the present with saying that we have written it by the side of the quotient, as we see in this example, that is to say, in writing the divisor under this remainder, and separating the one from the other by a line; and then we pronounce *five-sevenths*. We will hereafter explain the nature of these numbers.

61. If in the course of the operation, some of the partial dividends be found insufficient to contain the divisor, we write a cipher in the quotient, and omitting the multiplication, we immediately bring down another figure by the side of the partial dividend, and continue the division as before.

EXAMPLE.

It is required to divide 14464 by 8.

```
14464 | 8
  8   |----
------  1808
  64
  64
-----
   064
    64
   ---
```

I here take the two first figures of the dividend, because the first does not contain the divisor.

I find that 14 contains 8 once, I write 1 in the quotient; I multiply 8 by 1, and I subtract the product 8 from 14, which gives me 6 for the remainder, by the side of which I bring down the third figure 4 of the dividend.

I continue in saying, in 64 how many times 8? eight times;

I write 8 in the quotient and multiplying [illegible] in the product, which I take from the [illegible] remains 0, to the side of which I bring down [illegible] figure of the dividend; and as 6 does not contain [illegible] I write 1 in the quotient, and I bring down [illegible] the last figure of the dividend which is [illegible] and say, it is [illegible] many times 6? it is contained eight times [illegible] I in the quotient, I multiply and subtract the product [illegible] there remains nothing. I conclude that [illegible] contains [illegible] 1808 times.

Suppose that we have 858 to divide by 6

```
858 | 6
600 | 143
-----
 258
 240
-----
  18
  18
-----
```

Here it is evident that 6 is contained in 8 [illegible] than once, and consequently [illegible] in 800; it is also evident that 6 is not contained 100 times in 858, therefore the quotient will be between 10 and 100, and as all numbers containing [illegible] express them, there will be three figures in the quotient. Now having found that the first figure is 1, that is to say, 100, we multiply the divisor 6 by this 100, and we have 600 for the product; we subtract this product from the dividend and there remains 258 to be divided: now as the next figure in the quotient will be tens, and as the numbers 258 contain 25 tens, we seek how often 6 is contained in 25, we find 4 times and as this 4 is four tens, we place it on the right of the 1 [illegible], we multiply the divisor 6 by this 4 which gives 24 tens or 240, and subtracting 240 from 258 there remains 18 to be divided; then as 6 is contained in 18 three times, and as this 3 signifies 3 units, we place it on the right of the 4 tens and

multiplying the 6 by this 3, we have 18, which being subtracted from 18, there remains nothing.

Wherefore the division is complete, and we have 143 for the quotient, or the number of times that 6 is contained in 858.

We have pursued this method here merely for the sake of illustration; we shall see by performing the work in the usual way, that this is only a useless repetition of ciphers and of the figures in the dividend.

```
858 | 6
  6 | 143
 ----
  25
  24
 ----
   18
   18
 ----
   ..
```

Here the first figure 1 in the quotient, which I find in seeking how often 6 is contained in 8 hundred, signifies 1 hundred, and consequently the figure 6 which is the product of the divisor multiplied by this 1 is 6 hundred; having subtracted this 6 hundred from 8 hundred there remains 2 hundred, and as each hundred is ten tens, this 2 together with the 5 tens which I bring down is 25 tens, and the quotient figure 4 which I find in seeking how often 6 is contained in these 25 tens is 4 tens, and it receives its true value in being placed on the right of the 1 hundred. Again, multiplying 6 by 4 tens, I have 24 tens; this being placed under 25 tens and subtracted, leaves 1 ten, to the side of which bringing down the 8 units of the dividend, I have 18 units to be divided. I then say, 6 is contained in 18 three times; I place 3 in the quotient, which, as it is 3 units, stands on the right of the 4 tens, then multiplying I say, 3 times 6 are 18, which taken from the 18 remaining units of the dividend leaves nothing, and the quotient is 143 as before.

OF DIVIDING BY A NUMBER CONSISTING OF MANY FIGURES.

62. When the divisor consists of many figures, we shall proceed in the following manner:

Take on the left of the dividend as many figures as will contain the divisor. This done, instead of seeking as before how many times the part of the dividend which you have taken contains your whole divisor, seek only how many times the first figure of your divisor is contained in the first figure of your dividend, or in the two first, if the first is not sufficient; write the quotient under the divisor as before.

Multiply successively, according to the rule given (50,) all the figures of your divisor, by this quotient, and place the figures of the product under the corresponding figures of your partial dividend. Subtract, and to the side of the remainder bring down the next figure of the dividend, to continue the operation in the same manner.

We will elucidate this by some examples, and anticipate at the same time cases which might occasion some embarrassment.

EXAMPLE I.

We propose to divide 75347 by 53.

```
75347 | 53
53    | 1421 24/53
-----
 223
 212
 ---
  114
  106
  ---
   87
   53
   --
   34
```

I take only the two first figures of the dividend, because they contain the divisor, and instead of saying in 75 how many times 53, I seek only how often the 7 tens of 75 con-

tain the 5 tens of 53, that is to say, how many times 7 contains 5; I find once, wherefore I write 1 in the quotient.

I multiply 53 by 1, and I write the product 53 under 75: after the subtraction, there remains 22, to the side of which I bring down the figure 3 of the dividend, and I proceed, in saying, for greater facility, in 22 how many times 5, (instead of saying in 223 how many times 53;) I find 4 times, which I write in the quotient.

I multiply successively, by 4, the two figures of the divisor. and I write the product 212 under my partial dividend 223; after the subtraction, there remains 11; I bring down to the side of this remainder the figure 4 of the dividend, and I say simply, as before, in 11 how many times 5? twice; I write 2 in the quotient, and I multiply 53 by 2, which gives me 106, this I write under the partial dividend 114; subtracting, I have 8 for the remainder, to the side of which I bring down the last figure 7; I divide 87 in the same way, and continuing as before, I find 1 for the quotient, and 34 for the remainder. this I write on the side of the quotient, in the manner mentioned above (60.)

63. We ought, strictly speaking, to find how many times each partial dividend contains the whole divisor; but as this research would often be long and tedious, we content ourselves, as we have seen, with seeking how often the greater part of this dividend contains the greater part of the divisor. The quotient that we find by this method is not always the true quotient, because, in taking this part, we make, in reality, only an approximate estimation; but besides that this estimation almost always leads to the truth, and in the contrary case, varies but little, the multiplication which comes after serves to redress what has been defective in this judgment. In effect, if the partial dividend really contains the divisor 3 times, and if, by the trial which we make, we find that it contains it 4 times, we easily perceive that in multiplying by 4 we shall have a product greater than the dividend, since we shall take the divisor more times than it really is contained in this dividend, and consequently the subtraction will be impossible; we shall then diminish the quotient successively by one, two. &c. units, until we

find a product which we can subtract: on the contrary, if we place only 2 in the quotient, the remainder, after the subtraction, will be found greater than the divisor; which would prove that the divisor is yet contained in it, and that, consequently, the quotient is too small.

As for the rest, we acquire, in a little time, the habit of foreseeing by how many we ought to diminish or augment the quotient given by the first trial.

Example II.

We propose to divide 189492 by 375.

```
189492 | 375
1875   | 505 117/375
-------
  1992
  1875
  ----
   117
```

I take the four first figures of the dividend, because the three first do not contain the divisor.

After which, I say, in 18 only how many times 3? it is really contained 6 times; but in multiplying 375 by 6 I should have more than my dividend 1894; wherefore, I write only 5 in the quotient. I multiply 375 by 5; and having written the product under 1894, I subtract, and there remains 19.

I bring down by the side of 19 the figure 9 of the dividend: and as 199, which I then have, does not contain 375, I place 0 in the quotient, and I bring down by the side of 199 the figure 2 of the dividend, which gives me 1992, for which I say, in 19 only how many times 3? 6 times. But for the same reason as above, I write 5 only in the quotient; and having operated as before, I have 117 for remainder.

64. Observe here a reflection which may serve in a great many cases to prevent useless trials. We are principally exposed to these doubtful essays, when the second figure of the divisor (counting from the left) is sensibly greater than the first. In this case, instead of seeking how often the first figure of the divisor is contained in the corresponding part of the divi-

dend, we must seek how often this first figure, increased by a unit, is contained in the corresponding part of the dividend: this trial will always be much nearer than the first.

EXAMPLE.

We propose to divide 1832 by 288.

$$\begin{array}{r|l} 1832 & 288 \\ \cline{2-2} 1728 & 6\tfrac{104}{288} \\ \cline{1-1} 104 & \end{array}$$

Instead of saying, in 18 how many times 2? I shall say, in 18 how many times 3, because the divisor 288 is much nearer to 300 than to 200; I find 6, which is the true quotient; instead of which I should have found 9, and should, consequently, have been obliged to make three useless trials.

MEANS OF ABRIDGING THE PRECEDING METHOD.

65. It is to render the method more easy to be understood that we have prescribed to write under each partial dividend the product which we find in multiplying the divisor by the quotient; but as the end of arithmetic should be to shorten the operations, we think we ought here to remark that we may dispense with writing these products, by performing the subtraction regularly as we multiply each figure of the divisor, writing only the remainders. The following example will suffice to show how this subtraction is performed.

EXAMPLE.

We would divide 756984 by 932.

$$\begin{array}{r|l} 756984 & 932 \\ \cline{2-2} 1138 & 812\tfrac{200}{932} \\ 2064 & \\ 200 & \end{array}$$

Having taken the four first figures of the dividend, which are necessary to contain the divisor, I find that 75 contains 9, 8 times, I write 8 in the quotient; and instead of writing under 7569 the product of 932 by 8, I first multiply 2 by 8, which gives me 16; but as I cannot take 16 from 9,

I borrow from the next figure 6, a ten; which added to 9 gives 19, from this taking 16 there remains 3 which I write underneath.

To keep account of the ten borrowed, instead of diminishing by a unit, the figure 6 from which I have borrowed, I retain this unit which I shall add to the next product; thus continuing the multiplication, I say 8 times 3 are 24, and one which I have retained are 25; as I cannot take 25 from 6, I borrow from the next figure 6 in the dividend, two tens, which added to 6 gives me 26, from this I take 25 and there remains 1, which I write under 6; by this means I have kept account of the first ten which I should have taken from the 6, because I have subtracted one ten more. I retain in the same manner the two tens which I have borrowed. I then continue, saying 8 times 9 are 72, and 2 which I borrowed are 74, this taken from 75, leaves 1.

I bring down to the side of the remainder 113 the figure 8 of the dividend, and I proceed in the same manner, saying in 11 how many times 9? Once, after which once 2 is 2, this taken from 8 leaves 6; once 3 is 3, this taken from 3 leaves 0; once 9 is 9 which taken from 11 leaves 2. I bring down the figure 4 by the side of the remainder 206, and I say in 20 how many times 9? twice; and multiplying I say twice 2 are 4, which taken from 4 leaves 0; twice 3 are 6, which taken from 6 leaves 0; and lastly twice 9 are 18, which taken from 20 leaves 2.

66. It may happen in the course of these partial divisions, that the dividend may contain the divisor more than 9 times, we ought not however in any case to put more than 9 in the quotient, for if we could put 10, this would be a proof that the quotient found by the preceding operation was false, since the ten which we should find in the present quotient would belong to the first quotient.

67. When the dividend and divisor are both followed by ciphers, we can take as many from each as there are in that which contains the smallest number. For example, to divide 8000 by 400, I shall only divide 80 by 4; for it is evident that 80 hundreds will only contain 4 hundreds, as oft as 80 units contain 4 units.

OF THE DIVISION OF DECIMAL PARTS.

68. To avoid superfluous distinctions, we will reduce the division of decimals to this single rule.

Of the two proposed numbers, place on the right of that which has the fewest decimals as many ciphers as will render the decimal places in each equal; this will not alter the value of the number (30;) suppress the comma in both, and perform the operation as in whole numbers; the quotient will not be altered in any respect.

EXAMPLE.

We propose to divide 12,52 by 4,3.

I write - - - 12,52 | 4,3
or rather - - - 12,52 | 4,30

Suppressing the comma, I have 1252 to divide by 430; performing the operation,

1252 | 430
392 | $2\frac{392}{430}$

I find 2 for the quotient, and 392 for the remainder, that is to say, the quotient is 2 and $\frac{392}{430}$.

But as the object which we propose, when we make use of decimals, is to avoid ordinary fractions, instead of writing the remainder 392 in the form of a fraction, as above, we shall continue the operation as in the following example:

1252 | 430
2,9116+
3920
500
700
2700
120

Having found the quotient in whole numbers, which in this case is 2, we place on the right of the remainder 392 a cipher, which truly renders it ten times too great: we continue to divide by 430, and having found that we should place 9 in the quotient, we place it there in effect, but the place of the integers being marked in putting a comma after the

2; the 9 by this means only signifies tenths: having performed the multiplication and subtraction, we place a cipher on the right of the remainder 50, which is the same thing as if we had placed two on the right of the dividend; but in placing after 9 the quotient 1 which we find, we by this means give it its true value, since it then signifies a hundredth; we thus continue the operation as far as we think proper. In continuing to two decimals, we have the value of the quotient at least to less than a hundredth of a unit; carrying it to three figures, we have the quotient to less than a thousandth, and so on; since we could not put a unit more or less without rendering the quotient too great or too little. The sign + is placed on the right of the quotient to show that the operation might be extended farther, that is, that the quotient is somewhat less than the true quotient.

All remainders arising from divisions, may thus be reduced to decimals.

We have yet to explain why the suppression of the comma in the dividend and divisor does not alter the quotient, when we have rendered the number of decimals the same in each of these two numbers; this is easy to perceive, because in the above example, the dividend 12,52 and the divisor 4,30 are nothing else than 1252 hundredths and 430 hundredths, since the whole units are worth a hundred hundredths (22;) now it is evident that 1252 hundredths contain no otherwise 430 hundredths, than 1252 units contain 430 units; the consideration then of the comma is useless when we have completed the number of the decimals.

69. When we require the quotient of a division only to a proposed degree of exactness, we can contract the calculation by the following method. We will first suppose that we only require to have the quotient correct to less than a unit; afterward we will show how to apply the method, to have it as near as we would: observe the following rule.

Suppress, on the right of the dividend, as many figures, less one, as there are in the divisor; after which, perform the division as usual: if there be no remainder, you will place on the right of the quotient as many ciphers as you have suppressed figures in the dividend. But if there be a remainder,

you will continue to divide, not by the same divisor as before, which is no longer possible, but by the same divisor after having suppressed the last figure on the right; after this division, you will divide the new remainder by the preceding divisor, after having suppressed its right hand figure, and you will continue thus to divide, suppressing in each division one figure on the right of the divisor.

EXAMPLE.

We would have, to within a unit, the quotient of 8789236487 divided by 64423, I suppress the four last figures on the right of the dividend, and I divide 878923 by the proposed divisor 64423.

878923	64423
264693	136430
41424 . . .	6442
2772 . . .	644
196 . . .	64
4 . . .	6

I first find 13 for the quotient, and 41424 for the remainder: I then divide 41424 by 6442, suppressing the last figure 3 of the divisor. I have 6 for the quotient, which I place on the right of the first quotient 13, and the remainder is 2772, which I divide by 644; again suppressing one figure on the right of the primitive divisor, I have 4 for the quotient: this I place on the right of the principal quotient 136; the remainder is 196, which I divide by 64, again suppressing one figure of the divisor, the quotient is 3, and the remainder 4. Lastly, I divide by 6, and I have 0 for the quotient; so that the quotient of 8789236487 divided by 64423, is 136430, to within a unit. In effect, the exact quotient is $136430\frac{6297}{64423}$.

It is not necessary to write each time, as above, the new divisor; we may be satisfied with barring, in the primitive divisor, each figure as we proceed to a new division; we have written these divisors by the side of the successive remainders only to render the operation more clear.

70. If the remainder of the first division be found less than the divisor after having suppressed the right hand figure, we place a cipher in the quotient; and if it be yet less than the

divisor after having again suppressed the right hand figure, we again place a cipher in the quotient, and so on.

EXAMPLE.

To have within a unit, the quotient of 55106054 divided by 643, I divide, as usual, the part 551060 which remains after the suppression of the two last figures of the proposed dividend.

```
551060 | 643
  3666 | 85701
  4510
     9 . . . 64
     9 . . . 6
     3
```

I have 157 for the quotient, and 9 for the remainder: I must then divide this remainder by 64 only: as 9 does not contain the divisor, I put 0 in the quotient, and I have still 9 for the remainder; this I divide by 6 only, so that the quotient sought is 85701, which is within a unit.

71. If in the commencement of the operation, after having suppressed the figures on the right of the dividend as directed by the rule, the figures which remain do not contain the divisor, we suppress immediately as many figures on the right of the divisor as are necessary, in order that it may be contained in the dividend.

EXAMPLE.

We would have, within a unit, the quotient of 1611527 divided by 64524.

I suppress the four figures 1527 on the right of the dividend. But as the remaining figures 161 cannot be divided by 64524, I suppress in this divisor the three last figures 524, which must be suppressed in order that the divisor may be contained in the remaining dividend 161; therefore, I divide 161 by 64, operating as in the preceding example:

```
     | 64
161  | 25
33 . 6
3
```

and I have 25 for the quotient of 1611527, divided by 64524, to within a unit: in effect, the exact quotient is $24\frac{52951}{64524}$, which is much nearer to 25 than to 24.

72. As we suppress a figure in the divisor, it is requisite, for the sake of greater exactness, to augment by a unit the last of those which remain, if that which we suppress be equal to or greater than 5. We will augment in the same manner, by a unit, the last of the figures which remain in the dividend, after the suppression prescribed by the rule, if the figures suppressed surpass 5, 50, or 500, according as there may be 1, 2, or 3, etc.

Example.

We would have, to within a unit, the quotient of 3658627 divided by 1987.

I divide then 3658 by 1987, as follows:

	1987
8658	4357
710	199
113 .	20
13 .	2

that is to say, instead of dividing the remainder 710 by 198 only, I divide it by 199, because the last figure 7 which I suppress is greater than 5. The same reason for the following division. But as the last divisor 2 which is contained 6 times in 13, is a little too much, I place 7 in the quotient to compensate. It is now easy to see how we should proceed when we would have the quotient much more exact. For example, if we would have the quotient to within a ten-thousandth of a unit, the question would require as many ciphers (here it would require four) on the right of the dividend, as we would have decimals in the quotient: after which, we should perform the division according to the above method. When we shall have found the quotient to within a unit, we shall separate upon the right hand, by a comma, as many figures as we would have decimals.

Example.

We would have to within the ten-thousandth part of a unit, the quotient of 6927 divided by 4532: I place four ciphers on the right of 6927, and the question is reduced to this, viz.

to have the quotient of 69270000 divided by 4532, to within a unit, that is to say, conformably to the above rule, the quotient of 69270 divided by 4532. The operation is as follows:

```
69270 | 4532
23950 | 15285
 1290 . . 453
  384 . . 45
   24 . . 5
```

the quotient sought is therefore 1,5285, which is true to within a ten-thousandth of a unit.

If there were decimals in the dividend, or in the divisor, or in both, we should first transform them to whole numbers, according to what has been said (68;) after which we should operate as in this last example.

If then we would reduce a proposed fraction to decimals, we should come at it readily by this method, having regard to what has been said (71.)

Thus if we would reduce $\frac{4253}{9678}$ to decimals, and have the value of it to within the thousandth part of a unit, we shall have 4253000 to divide by 9678; which (69) will be reduced to the division of 4253 by 9678, and (71) to the division of 4253 by 968, according to the above method. We shall then find 439; so that we shall have 0,439 for the value of $\frac{4253}{9678}$, to within a thousandth.

The following may serve as an illustration of this method of contracting a division:

If we have 2264385 to divide by 2264, by the rule (69) we shall have 2264 to divide by 2264, and consequently the quotient will be 1, with as many ciphers as there were figures suppressed in the dividend, that is 1000.

Now the true quotient is 1000 $\frac{385}{2264}$, whence it may easily be seen that as the number of figures suppressed, is always one less than the number of figures in the divisor, the quotient arising from this part of the dividend can never equal a unit.

With regard to the number of figures which would arise in the quotient, from the bringing down of these figures: as we suppress in the divisor one figure before we commence the

division of the remainder, when there is one, and as we put a figure in the quotient for each figure which we afterward suppress in the divisor, it is plain that this will produce the same number of figures in the quotient, as when we divide in the common way.

TO DIVIDE A NUMBER BY A SINGLE FIGURE.

Suppose we have 8645 to divide by 4, I place the numbers thus:

Dividend
Divisor 4 | 8645

2161¼ Quotient.

After which I say: 4 is contained in 8 twice, I place 2 under 8, then 4 is contained in 6 once, and 2 remains; I write 1 under 6 and the remainder two together with the 4 on the right is 24, then as 4 is contained in 24 six times, I write 6 under 4; lastly 4 is contained in 5 once; I write 1 under 5, and as there is a remainder of 1, I write ¼ on the right, and I have 2161¼ for the quotient of 8645 divided by 4. The same method for any other figure.

TO DIVIDE ANY NUMBER BY FIVE, TWENTY-FIVE, etc.

Separate by a comma one figure on the right of the given dividend, and multiply the whole by 2. If we have 82631 to divide by 5, I separate the right-hand figure of the dividend thus: 8263,1

2

16526,2

and having multiplied by 2, I have 16526,2, which is twice one-tenth of the given dividend, or one-fifth as was required. Performing the same operation as in the preceding example, thus:

5 | 82631

16526⅕

I have the same quotient.

Also, if I have 743256 to divide by 25, I separate two figures on the right of the given dividend, and then multiply by 4, thus

$$\begin{array}{r} 7432{,}56 \\ 4 \\ \hline 29730{,}24 \end{array}$$

where having taken 4 times a hundredth part, I have a twenty-fifth part. To multiply any number by 5. Place a cipher on the right of a given number, and divide by 2. If we have 376023 to multiply by 5. I operate thus:

$$\begin{array}{r|l} 2 & 3760230 \\ \hline & 1880115 \end{array}$$

where having taken the half of ten times the given number, I have five times, as was required. To multiply by 25. Place two ciphers on the right of the given number, and divide by 4. Thus, if we have 896453 to multiply by 25, we may operate as follows:

$$\begin{array}{r|l} 4 & 89645300 \\ \hline & 22411325 \end{array}$$

where having taken a fourth part of a hundred times, I have 25 times.

÷ *By*, is the sign of division, and signifies that the former of the two quantities between which it is placed, is to be divided by the latter. Thus, 24÷6, shows that 24 is to be divided by 6, and is read twenty-four by 6, or twenty-four divided by six. The same thing is signified by the numbers being placed thus, $\frac{24}{6}$.

Examples for Practice.

1. 24÷6=4, or $\frac{24}{6}=4$, either of which is read twenty-four by six equal to four.
2. 5360,7÷83,5=64,2.
3. 2156,36÷0,235=9176.
4. 2634÷0,5=5268.
5. 739÷0.25=2956.
6. 739÷2956, or $\frac{739}{2956}=0{,}25$. (73)

7. $353 \div 2824$, or $\frac{353}{2824} = 0,125$.

8. $1 \div 0,125 = 8$, and $353 \div 0,125 = 2824$.

10. $24 \times 3 \div 3 = 24$. By this we see that the signs $\times$ and $\div$, that is, that the operations of multiplication and division destroy the effect of each other: wherefore, as any number, having been multiplied and divided by the same number, is not altered, we may, in all such cases, spare ourselves the trouble of the operation.

11. $24 \times 9 \div 3 = 72$. Here we may remark, that as 9, the number by which we mutliply, is just three times the number 3, by which we divide, the result 72 is just 3 times 24. Also if either of the numbers 24 or 9 be first divided by 3 and the quotient multiplied by the remaining number, the result will be the same: thus, $24 \div 3 = 8$, and $8 \times 9 = 72$; again, $9 \div 3 = 3$, and $24 \times 3 = 72$, which method of operating, when practicable, is always the shortest.

Or thus, $9 = 3 \times 3$, therefore, $24 \times 9 \div 3 = 24 \times 3 \times 3 \div 3 = 24 \times 3 = 72$, because $\times 3 \div 3$ destroy each other, as in the preceding example.

If there be any two numbers, suppose 4 and 2, and if each of these be multiplied by the same number, suppose 3, then the two products 12 and 6 are multiples of the two numbers 4 and 2, also because the multiple 12 contains 4 as often as the multiple 6 contains 2; the numbers 12 and 6 are called equimultiples of the numbers 4 and 2.

Now, it may be observed of these two numbers, and the two equimultiples, that 12 contains 6 as often as 4 contains 2, that is, that 3 times 4 contains 3 times 2 as often as 4 contains 2. The same would be the case with any other numbers and any other equimultiples of those numbers.

12. $8 \times 24 \div 48 = 4$. Here, as the number 48 can be divided exactly by 24 or by 8, we may divide by either of them, and then divide the other by the quotient, the result will still be 4. Thus, $48 \div 24 = 2$, and $8 \div 2 = 4$, or $48 \div 8 = 6$, and $\frac{24}{6} = 4$. Or thus, as $48 = 24 \times 2$, we have 8×24 to divide by 2×24, and, according to what has been said above, as 24 times 8 contains 24 times 2 as often as 8 contains 2, $\frac{24 \times 8}{24 \times 2} = \frac{8}{2} = 4$; or, as $48 = 6 \times 8$, we have $\frac{8 \times 24}{8 \times 6} = \frac{24}{6} = 4$.

If a number divide the whole of another number and one of its parts, it will also divide the other part. Thus, 4 divides 32; now, if any part be taken from 32, which is a multiple of 4, the remaining part will also be a multiple of 4, which will more clearly appear as follows: $32=4+4+4+4+4+4+4+4$; and if, as by the supposition, we take away a certain number of these fours, the remainder will be a certain number of fours. Wherefore, if a number, &c.

PROOF OF MULTIPLICATION AND DIVISION.

74. We can draw from the definition which we have given of each of these two operations, the method of proving it.

Since, in multiplication, we take the multiplicand as many times as the multiplier contains a unit, it follows that if we seek how many times the product contains the multiplicand, that is to say, (59,) if we divide the product by the multiplicand, we shall have the multiplier for the quotient; and as we can take the multiplicand for the multiplier, and *vice versa;* in general, *if we divide the product of a multiplication by one of its factors, we shall have the other factor for the quotient.*

For example, having found above (50) that 2864 multiplied by 6 gives 17184, I divide 17184 by 2864; I ought to find, and I find, in effect, 6 for the quotient.

In like manner, since the quotient of a division shows how many times the dividend contains the divisor, it follows, that if we take the divisor as many times as it is signified by the quotient, we shall reproduce the dividend, if the division has been made without a remainder; and that, in the case where there is a remainder, if we multiply the divisor by the quotient, and if to the product we add the remainder of the division, we shall reproduce the dividend.

For example, we have found (63) that 189492 divided by 375 gives 505 for the quotient, and 117 for the remainder. In multiplying 375 by 505, we find 189375, to which adding the remainder 117, we again find the dividend 189492.

Thus, multiplication and division may serve to prove one another reciprocally.

But we can verify these operations by a shorter method, which we are about to explain: we must not, however, on this

account neglect the reflections which we have made, as they will be useful on many other occasions.

PROOF BY NINE.

75. Suppose that having multiplied 65498 by 454, and found that the product is 29736092, we would prove whether the work be right.

We add all the figures 6,5,4,9,8 of the multiplicand, as if they only contained simple units, and subtracting 9 as often as it is found in the sum, we have 5 for the remainder.

We also add the figures 4,5,4, of the multiplier, and subtracting in like manner all the nines that this addition produces, we have 4 for the remainder.

We multiply the remainder 5 of the multiplicand by the remainder 4 of the multiplier, and from the product 20 we subtract the nines which it contains: there remains 2.

If the product be exact, in adding all the figures 2,9,7,3,6, 0,9,2, of this product, and subtracting all the nines, there should remain only 2; which in effect is the case.

This rule is founded upon the principle that, to have the remainder of the subtraction of all the nines that a number contains, we have only to find the remainder that these figures, added as simple units, would give after the suppression of the nines.

In effect, if from a number expressed by a single figure followed by several ciphers, we subtract all the nines, the remainder will be expressed by that single figure. If from 4000, 500, or 60000 you subtract all the nines, the remainder will be 4, 5, or 6, etc. which is easy to perceive.

The remainder then which we should find after the suppression of the nines from a number, such as 65498 (which is the same as 60000+5000+400+90+8,) will be the same as that which we should find from 6+5+4+9+8; that is to say, the same as if we added these figures as containing simple units.

The application of this principle to the proof of the multiplication, is as follows:

Since 65498 is composed of a certain number of nines and a remainder of 5, and the multiplier 454 of a certain num-

ber of nines and a remainder of 4, it can only be from the product of 5×4, or 20, that the total product is not divisible by 9: there ought then to remain in the product of the two given numbers the same quantity as in the product of the two remainders, after the suppression of the nines in each.

We could also perform this proof in the same manner by the number 3.

With regard to division, it is easy to prove the operation, from what has been said (70.)

Having taken from the dividend the remainder given by the division, we shall regard the result as a product of which the divisor and quotient are factors, and consequently we shall apply the proof by 9, in the same manner as above.

Strictly speaking, this verification is not infallible; for if in the multiplication, for example, we had mistaken in some figure of the product, and if we had made an equal error, but in a contrary sense, in some other figure of the same product; as that would not at all alter the remainder that we should have after the suppression of the nines, this rule would not discover the error; but as there must be, as we see, at least two errors, and two errors which balance, or at least which differ only by a certain number of times 9, the cases in which this verification would be faulty, will be very rare in practice.

SOME USES OF THE PRECEDING RULE.

76. Division serves not only to find how many times one number contains another, but also to divide a number into equal parts.

To take the half, the third, the fourth, the fifth, the twentieth, the thirtieth, etc. of a number, is to divide this number by 2, 3, 4, 5, 20, 30, etc., or to separate it into 2, 3, 4, 5, 20, 30, etc. equal parts, in order to take one of those parts.

Among many examples of this use of division, we choose the case in which we would find a mean quantity among several others. Suppose that we would dig a canal through a certain tract of land, and that having taken the depth of cutting at ten different stations we have the ten following depths:

Stations.	Depth of Cutting.
1 - - -	3,50 feet.
2 - - -	3,00
3 - - -	5,40
4 - - -	6,25
5 - - -	10,75
6 - - -	11,10
7 - - -	10,00
8 - - -	8,42
9 - - -	5,18
10 - - -	5,40
Sum of the Cuttings, -	69,00
Average or Mean Cutting, -	6,9

What we understand by a mean quantity is what each quantity would be, if (their total value remaining the same,) the quantities were all equal. It is plain that if they were all equal, to have the value of each, we should have to divide their total sum into as many parts as there are quantities. We must then divide by ten: the quotient 6,9 is therefore the quantity or mean depth, which we style thus because it holds a species of middle between the others.

Division serves also to convert the units of a certain kind into units of a superiour kind; for example, a certain number of pence into shillings, and these into pounds. To reduce 5864 pence into shillings, we may remark, that since it requires 12 pence to make one shilling, as many times as there shall be 12 pence in 5864 pence, there will be so many shillings; we must then divide by 12, and we shall find 488 shillings and 8 pence remaining. To reduce into pounds the 488 shillings, we shall divide 488 by 20, since there must be 20 shillings to make 1 pound, and we shall have in all 24 pounds 8 shillings and 8 pence.

With regard to the dividing by 20, let us observe that when we have to divide by a number followed by ciphers, we can abridge the operation by separating on the right of the dividend as many figures as there are ciphers; we divide the part which remains on the left by the significant figures of the divisor, and if there be a remainder, we place on the right of it the figures which we have separated; this will give the total

remainder. For example, to divide 5834 by 20, I separate the last figure 4, and I divide by 2 the remaining part 583; I have 291 for the quotient, and 1 for the remainder; I place on the right of this remainder 1 the figure 4, which gives me 14 for the whole remainder, so that the quotient is $291\frac{14}{20}$.

When we would have in pounds and shillings the twentieth of a proposed number of pounds, we shall consider the last figure as so many shillings, and take half of the other figures which we shall consider pounds. If in taking this half there remains a unit, we shall count it for ten shillings in placing it on the left of the figure we first separated. For example, if we would have the twentieth of 54672 pounds, we shall separate the last figure 2, which we shall count for 2 shillings; and taking the half of 5467, which is 2733, with a unit remaining, we shall write 2733 pounds 12 shillings: the reason of this rule is evident, from the consideration that 54672*l.* is 54660 pounds plus 12 pounds; now the twentieth of 54660 is evidently 2733, and that of 12 pounds is 12 shillings, since the twentieth of a pound is a shilling. If there were shillings and pence in the proposed sum, we should neglect the pence, of which the twentieth part can never make a penny. With regard to the shillings, we should triple them; and taking the fifth, we should consider the quotient as pence. Thus the twentieth of 54672*l.* 17*s.* 7*d.* is 2733*l.* 12*s.* 10*d.* This multiplying the shillings by 3 and dividing by 5, instead of multiplying by 12 and dividing by 20, is done upon the principle that $\frac{12}{20}=\frac{3\times4}{5\times4}=\frac{3}{5}$ which will be understood from what has been said above on the equimultiples of numbers.

TO FIND THE LEAST COMMON MULTIPLE OF SEVERAL NUMBERS.

If no number can be found that will divide two of the given numbers without a remainder, multiply them all together, the product will be the least common multiple.

When a number can be found that will thus divide two or more of them, divide by that number, then divide the quotients and those of the given numbers, if there be any, which could not be divided by the first divisor, by any number that will

divide two or more of them; proceed in this manner till no number can be found that will any longer divide two or more, after which multiply all the divisors and remaining quotients, if there be any, the product will be the least common multiple sought.

EXAMPLE.

Suppose that we would find the least common multiple of the numbers 2,6,7,14: I place the numbers as follows:

$$\begin{array}{r|l} 2 & 2,6,7,14 \\ \hline 7 & 1,3,7,\ 7 \\ \hline & 1,3,1,\ 1 \end{array} \qquad 2\times7\times3=42$$

and having found that 2 will divide 2, 6, and 14, I therefore divide these numbers by 2, and I have the quotients 1, 3, and 7, which I place respectively under the figures divided: also, as 2 will not divide 7, I place it under the line together with the quotients found. I then seek in the suite 1,3,7,7 for two numbers which can both be divided by the same number without a remainder, and finding two sevens, I therefore divide by 7, and I have the suite 1,3,1,1, which can no longer be divided. I then multiply the two divisors and the remaining number 3, which gives me 42 for the least common multiple of the numbers 2, 6, 7 and 14. The three numbers 1,1,1 are omitted, because the multiplication of them would not alter the product.

It is evident in the above example that if we multiply the first suite of quotients by 2, we shall reproduce the numbers divided; also, because the 7 which was not divided is retained, the product of the divisor 2 and the numbers 1,3,7,7 will be a multiple of all the given numbers. Again, for the same reason, the product of the last divisor 7, and the numbers 1,3,1,1, will be a multiple of the numbers 1,3,7,7; also, from what has been said of equimultiples, it is plain that if 21 be a multiple of any number, twice 21 will be the same multiple of twice that number. Therefore, the product 42 is a common multiple of all the given numbers.

Also, 42 is the least common multiple of the numbers 2,6,7,14, for if not, let 41 be the least common multiple.

Then by this supposition, 14 divides 41. Now $41 = 28 + 13$: then, because 14 divides 41, and the part 28, and because any number which divides a number and one of its parts will also divide the other part, 14 will therefore divide 13, which is absurd: wherefore 42 is the least common multiple.

In the same manner, 84 is found to be the least common multiple of 3, 14, 21 and 28.

OF FRACTIONS.

77. Fractions, arithmetically considered, are numbers by which we express quantities less than a unit.

78. To form a just idea of fractions, we must conceive that the quantity first taken for the unit is itself composed of a certain number of smaller units; as we conceive, for example, that the pound is composed of twenty smaller parts, called shillings.

One or more of these parts forms what is called a fraction of a unit. We also give this name to the numbers which represent these parts.

79. A fraction may be expressed in numbers, in two ways, which are both in use.

The first consists in representing, as whole numbers, the parts of a unit which contain the quantity in question; but then we give a particular name to these parts: thus, to express 7 parts of which we conceive 20 in the pound, we should employ the figure 7, but we should pronounce 7 shillings, and we should write 7*s*. This method of signifying the parts of a unit is used in compound numbers, of which we shall speak hereafter.

80. But as this would require a particular sign for each division which we could make of the unit, we avoid this multiplicity of signs in expressing a fraction by two numbers placed one under the other, and separated by a line. Thus, to signify the 7 parts in question, we write $\frac{7}{20}$; that is to say, in general, we first write the number which signifies how many parts of a unit the quantity in question contains, and we write underneath this number that which signifies how many of these parts we conceive the unit to contain.

81. And to read a fraction, we first read the upper number, which is called *the numerator;* afterward the lower number, which is called *the denominator;* but we add to the name of this last the termination *ths* or *eths:* for example, to read $\frac{7}{25}$ we pronounce *seven-twentieths;* to read $\frac{4}{5}$ we pronounce *four-fifths*, we should here understand four parts of which there must be five to compose a unit.

We must except from the general termination, the fractions whose denominator is 2 or 3, which are pronounced halves, thirds. Thus the fractions $\frac{1}{2}$, $\frac{2}{3}$ are pronounced *one-half*, *two-thirds*.

82. The numerator then shows how many parts of a unit the quantity represented by the fraction contains; and the denominator makes known the value of these parts, in signifying how many such parts there must be to compose a unit. We give it the name of denominator, because it is in effect the denominator which gives the name to the fraction, and which in these two fractions, for example, $\frac{3}{5}$ and $\frac{2}{7}$, gives the name of *fifths* to the parts contained in the first, and the name of *sevenths* to those contained in the second.

83. The numerator and denominator, are called also by a common name, the *two terms of the fraction.*

OF WHOLE NUMBERS CONSIDERED UNDER THE FORM OF FRACTIONS.

84. The operations which we make upon fractions often lead to fractional results, of which the numerator is greater than the denominator, for example, to results such as $\frac{13}{8}$, $\frac{27}{5}$, etc.

These expressions are not properly called fractions, but they are whole numbers united to fractions.

85. To extract the whole numbers which are contained in them, we must divide the numerator by the denominator. The quotient will signify the whole units, and the remainder will be the numerator of the fraction which accompanies these units. Thus $\frac{27}{5}$ will give $5\frac{2}{5}$, that is to say, five whole units, and two-fifths of a unit.

In effect, in the expression $\frac{27}{5}$, the denominator 5 shows that the unit is composed of 5 parts: as many times then as

there shall be 5 in 27, so many whole units there will be in the value of the fraction $\frac{27}{5}$.

Let us here recollect that the expressions $\frac{27}{5}$ and $27 \div 5$ are the same.

We have seen in multiplication, that the product of a number by another number is equal to the sum of the products when all its parts are multiplied by that number, whatever those parts may be: thus, for example, $146 = 100 + 40 + 6$, and 146×6 is therefore equal to $\overline{100 + 40 + 6} \times 6 = 600 + 240 + 36$ or 876.

Now it is evident, since $146 \times 6 = 876$, that if we divide 876 by 6 we shall have 146 for the quotient: but instead of dividing 876 by 6, we will divide, severally, each of the parts 600, 240, and 36, of which it is composed, by 6, and we shall have the three quotients 100, 40, and 6, now the sum of these quotients is 146: therefore when any number is divided by another number the quotient is equal to the sum of the quotients when all its parts are divided separately by that number, whatever the parts may be into which the number is divided.

Hence if we had the expression $\frac{600}{6} + \frac{240}{6} + \frac{36}{6}$, because the sum of the three quotients, when each numerator is divided separately by 6, is equal to the quotient when the sum 876 of the numerators is divided by 6, we first add all the numerators 600, 240, and 36 together, and divide the whole by 6. Also if we do not choose to perform the division we can leave it in the form of a fraction, thus

$$\frac{600}{6} + \frac{240}{6} + \frac{36}{6} = \frac{600 + 240 + 36}{6} = \frac{876}{6}.$$

86. The multiplications and divisions of whole numbers connected with fractions, require, at least for the sake of facility, that we should convert these whole numbers to fractions.

We make this conversion in multiplying the whole number, by the denominator of the fraction to which we would reduce this whole number. For example, if we would convert 8 whole units to fifths, we multiply 8 by 5, and we have $\frac{40}{5}$. In effect, when we would convert 8 into fifths, we

regard the unit as being composed of 5 parts; the 8 units will then contain 40 of these parts: in like manner $7\frac{4}{9}$, converted into ninths, makes $\frac{67}{9}$, because the 7 converted into ninths is $\frac{63}{9}$ and $\frac{63}{9}+\frac{4}{9}=\frac{63+4}{9}=\frac{67}{9}$. Also $13\frac{2}{3}=\frac{41}{3}$; $11\frac{3}{5}=\frac{58}{5}$; $23\frac{7}{8}=\frac{191}{8}$; $103\frac{11}{12}=\frac{1247}{12}$ and $341\frac{9}{11}=\frac{3760}{11}$.

OF THE CHANGES WHICH MAY BE MADE IN THE TWO TERMS OF A FRACTION, WITHOUT ALTERING ITS VALUE.

37. It is plain, that the more we conceive of parts in the unit, the more there must be of these parts to compose a certain quantity.

38. We may then render the denominator of a fraction double, triple, quadruple, etc. without in the least changing the value of the fraction, provided that at the same time we also render the numerator double, triple, quadruple, etc.

We therefore say, in general, that *a fraction is not altered in its value when both its terms are multiplied by the same number.*

Thus, $\frac{3}{4}$ is the same as $\frac{6}{8}$; $\frac{1}{2}$ the same thing as $\frac{2}{4}$, $\frac{3}{6}$, $\frac{5}{10}$, etc.

39. By a similar reasoning, we see that the fewer parts we conceive in the unit, the fewer of these are required to compose a certain quantity; and, consequently, we can, without changing the value of a fraction, render its denominator 2, 3, 4, etc. times less, provided that at the same time we render its numerator 2, 3, 4, etc. times less; therefore, in general, *a fraction is not altered in value when we divide its two terms by the same number.*

To see distinctly the truth of these two propositions, it is sufficient to recollect what is the numerator and the denominator of a fraction.

Let us then remark, that to multiply or divide the two terms of a fraction by the same number is not to multiply or divide the fraction, since, as we have said, this does not alter its value.

The two principles that we have established are the basis of the two following reductions, which are of great use.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR.

90. 1st. To reduce two fractions to a common denominator, multiply the two terms of the first, each by the denominator of the second, and the two terms of the second each by the denominator of the first.

For example, to reduce to a common denominator the two fractions $\frac{2}{3}$, $\frac{3}{4}$, I multiply 2 and 3, which are the two terms of the first fraction, each by 4, denominator of the second, and I have $\frac{8}{12}$, which (88) is of the same value as $\frac{2}{3}$.

I also multiply the two terms 3 and 4, of the second fraction, each by 3, denominator of the first, and I have $\frac{9}{12}$ which is of the same value as $\frac{3}{4}$; so that the fractions $\frac{2}{3}$ and $\frac{3}{4}$ are changed to $\frac{8}{12}$ and $\frac{9}{12}$, which are, respectively, of the same value as those, and which have each the same denominator.

It is easy to see that by this method the denominator will always be the same for each of the two new fractions, since in each operation the new denominator is formed by the multiplication of the two primitive denominators.

91. 2d. If we have more than two fractions, we shall reduce them all to the same denominator, in multiplying the two terms of each by the product resulting from the multiplication of the denominators of the other fractions.

For example, to reduce to a common denominator the four fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{7}$, I shall multiply the two terms 2 and 3 of the first, by the product of the three denominators 4, 5, 7, of the other fractions, which product I find in saying 4 times 5 are 20, and 7 times 20 are 140; I then multiply 2 and 3 each by 140, and I have $\frac{280}{420}$, which is of the same value as $\frac{2}{3}$ (88.)

In like manner, I multiply the two terms 3 and 4, of the second fraction by the product of 3, 5, 7, which product I form in saying 3 times 5 are 15, then 7 times 15 are 105; I then multiply 3 and 4 each by 105, which gives me $\frac{315}{420}$, a fraction of the same value as $\frac{3}{4}$.

Passing to the third fraction, I multiply its two terms 4 and 5 each by 84, the product of the three denominators 3, 4, and 7, and I have $\frac{336}{420}$, instead of $\frac{4}{5}$.

Lastly, for the fourth, I multiply 5 and 7 each by the product 60 of the denominators 3, 4, 5, of the three first frac-

tions, and I have $\frac{300}{420}$, instead of $\frac{5}{7}$; so that the four fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{7}$, are changed to $\frac{280}{420}$, $\frac{315}{420}$, $\frac{336}{420}$, $\frac{300}{420}$, less simple than those, it is true, but of the same value, and more susceptible, by their common denominator, of the operations of addition and subtraction.

Let us observe, that as the denominator of each new fraction is formed of the product of all the primitive denominators, this new denominator cannot fail to be the same for each fraction.

By the use of the sign $\times$, the above operation is written concisely thus:

$3\times4\times5\times7=420$, the general denominator.

Then, for each respective numerator,

$2\times4\times5\times7=280$, the first new numerator, or that of $\frac{2}{3}$,
$3\times3\times5\times7=315$, second - - - - - - $\frac{3}{4}$,
$4\times4\times3\times7=336$, third - - - - - - $\frac{4}{5}$,
$5\times5\times4\times3=300$, fourth - - - - - - $\frac{5}{7}$,

and, placing each new numerator over the common denominator, the fractions are $\frac{280}{420}$, $\frac{315}{420}$, $\frac{336}{420}$, and $\frac{300}{420}$.

This rule may be presented under another form, which leads to the giving a more simple expression to fractions reduced to a common denominator, when their primitive denominators are multiples one of another, or when they have a common measure.—Note. A common measure of two or more numbers is a number that will divide each of them without leaving a remainder; thus, 3 is a common measure of the numbers 6, 9, and 12.

We will take for a common denominator the least number which is exactly divisible by each of the denominators of the proposed fractious; that is, we shall take the least common multiple of these denominators, the method for finding which has been given above; and to have the numerator, which, for each fraction, will comport with this new denominator, we will multiply the given numerator by the number which shows how often its denominator is contained in the common denominator.

For example, if I have the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{3}{8}$, $\frac{7}{12}$, to reduce to the same denominator. I take the least common multiple of all the denominators for a common denominator,

which, in this case, is 24, and as 24 contains the denominators 3, 4, 6, 8, 12, as many times as are expressed by the following numbers 8, 6, 4, 3, 2, I write, as is here seen, these numbers each under its corresponding fraction.

$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{3}{8}$	$\frac{7}{12}$	24 Common denominator, or least common multiple.
8	6	4	3	2	

And multiplying each numerator by the corresponding figure in the lower line, I have $\frac{16}{24}$, $\frac{18}{24}$, $\frac{20}{24}$, $\frac{9}{24}$, $\frac{14}{24}$, for the fractions, reduced to the most simple common denominator.

In the same manner, we find that $\frac{1}{4}$, $\frac{2}{3}$, and $\frac{3}{4}$ are equivalent to $\frac{15}{60}$, $\frac{40}{60}$, and $\frac{36}{60}$; that $\frac{4}{5}$, $\frac{4}{9}$, and $\frac{6}{7}$ are equivalent to $\frac{252}{315}$, $\frac{140}{315}$, and $\frac{270}{315}$; and, lastly, that $\frac{3}{4}$, $\frac{5}{6}$, $\frac{6}{7}$, and $\frac{7}{11}$ are equivalent to $\frac{20772}{27772}$, $\frac{18540}{27772}$, $\frac{23376}{27772}$, and $\frac{17654}{27772}$.

REDUCTION OF FRACTIONS TO THEIR MOST SIMPLE EXPRESSION.

92. A fraction is so much the more simple when its two terms are reduced to smaller numbers. It is often possible to express a proposed fraction by smaller numbers, and that is the case when each of its terms can be divided by the same number; as this operation does not alter its value, (89) it is a simplification which we ought not to neglect.

The rule is as follows:

93. We divide the numerator and the denominator each by 2, and we repeat this division as often as it can be performed exactly.

We then divide the two terms by 3, and we continue to divide each by 3, as often as this can be done.

We do the same thing successively with the numbers 5, 7, 11, 13, 17, etc.; that is to say, with the numbers which have no divisor but themselves, or a unit, which we call *prime numbers*.

The only difficulty in this case is to know when we can divide by 2, 3, 5, etc.

We can, in this research, aid ourselves by the following principles.

94. Every number which terminates with a cipher is divisible by 2.

Every number, of which the sum of all the figures, added together, as if they were simple units, is 3, or a multiple of 3, that is to say, an exact number of times 3, is divisible by 3. For example, 54231 is divisible by 3, because these figures 5, 4, 2, 3, 1, make 15, which is five times three.

The same is the case with the number, if the figures added together make 9, or a multiple of 9.

* The demonstration of this proposition is as follows:

We can divide by 9 or by 3 any number of which the sum of the figures is divisible by 9 or by 3.

Let the number be 18765, of which the sum of the figures is 27: 27 being divisible by 9 and by 3, I say that 18765 is divisible by 9 and by 3.

For $18765=10000+8000+700+60+5$;

$10000=9999+1$;

$8000=8\times1000$, that is to say $(999+1)8$, that is to say $999\times8+8$;

$700=100\times7$, that is to say, $(99+1)7$, that is to say $99\times7+7$;

$60=10\times6$, that is to say $(9\times1)6$ or $9\times6+6$:

The number 18765 is therefore equal to the sum of all the following numbers:

$$\begin{array}{r} 9999 + 1 \\ 999\times8+8 \\ 99\times7+7 \\ 9\times6+6 \\ +5 \end{array}$$

But each of the numbers 999×8; 99×7; 99×6; is divisible by 9 and by 3; and the sum of the numbers 1, 8, 7, 6, 5, which is 27, is also divisible by 9 and by 3; therefore, the sum of all these numbers is divisible by 9 and by 3, for if a number divide all the parts of another, it also divides that other number. Wherefore the number 18765 is divisible by 9 and by 3. Therefore, etc.

Every number terminated by 5 or by a cipher, is divisible by 5.

With regard to the number 7 and those which follow, though it be easy to find similar rules, as the search which

they require is as long as the division, we should try the division.

Let us propose, for example, to reduce the fraction $\frac{2016}{5706}$ to its lowest terms. I divide the two terms by 2, because the two last figures of each are even, and I have $\frac{1008}{2853}$. I divide again by 2, and I have $\frac{504}{1449}$.

What has been said above shows me that I can divide by 3; I divide and I have $\frac{168}{483}$; I divide again by 3, which gives me $\frac{56}{161}$; lastly, I try to divide by 7, the division succeeds, and gives me $\frac{8}{23}$, which can no longer be reduced.

The reason why we prescribe to try the division only by the prime numbers 2, 3, 5, 7, etc. is that after having exhausted the division by 2, for example, it is useless to try the division by 4, for if this could succeed, much more should the division by 2 still succeed.

95. Of all the methods which we can employ to reduce a fraction to a more simple expression, the most direct is that of dividing the two terms by the greatest common divisor: the rule for finding which is as follows:

Divide the greater of the two terms by the less, if there be no remainder, the least term is the greatest common divisor.

If there be a remainder, divide the least term by this remainder, and if the division leave no remainder, the first remainder is the greatest common divisor.

If the second division give a remainder, divide the first remainder by the second, and continue always to divide the preceding remainder by the last remainder, till you come to an exact division.

The last divisor that you have employed will be the greatest common divisor, or greatest common measure of the two terms of the fraction.

If the last divisor be a unit, it is a proof that the fraction cannot be reduced.

Let us take for example the fraction $\frac{3760}{9024}$.

I divide 9024 by 3760; I have 2 for the quotient, and 1504 for the remainder.

I divide 3760 by 1504; I have 2 for the quotient, and 752 for the remainder.

I divide the first remainder 1504 by the second remainder 752; the division is exact, and I conclude that 752 will divide the two terms of the fraction $\frac{3760}{9024}$, and reduce it to its most simple expression, which, in performing the operation, I find to be $\frac{5}{12}$.

In effect, we have found that 752 divides 1504; it ought therefore to divide 3760, which, as we have seen, is composed of twice 1504 plus 752: we see also that it ought to divide 9024, since 9024 is composed of twice 3760 plus 1504.

We see besides that 752 is the greatest common divisor that 3760 and 9024 can have; for there cannot be a common divisor between 9024 and 3760, which is not at the same time the common divisor of 3760 and 1504; and between these two there cannot be one which is not at the same time the common divisor of 1504 and 752; but it is evident that between these two there cannot be a greater common divisor than 752: Wherefore, etc.

* OTHERWISE

To find the greatest common divisor of two numbers. Let the numbers, for example, be the two numbers 9024 and 3760.

Divide 9024 by 3760; the quotient will be 2, and the first remainder 1504.

Divide 3760 by the first remainder 1504; the quotient will be 2, and the second remainder 752.

Divide the first remainder 1504 by the second remainder 752, the quotient will be 2 without remainder.

I say first that 752 is a common divisor of the two numbers 9024, 3760.

For, since 752 divides 1504, 752 will divide twice 1504, because if a number divides another, it also divides all its multiples. But 752 divides 752, and 752 divides 1504; therefore 752 divides twice 1504 plus 752, that is to say, 3760; because if a number divides the two parts of another number, it also divides that other number.

Since 752 divides 3760, 752 will divide twice 3760. But 752 divides 1504; therefore 752 will divide twice 3760 plus

1504, that is to say, 9024; wherefore 752 is a common divisor of the two numbers 9024 and 3760.

I say also that 752 is the greatest common divisor of the two numbers 9024 and 3760.

For if 752 be not the greatest common divisor of the two numbers 9024 and 3760, a number greater than 752 will be the greatest divisor of these two numbers. Let it be for example 753.

Since 753 divides 3760, 753 will also divide twice 3760.

But it divides 9024; therefore 753 will divide 1504, because if a number divides another and one of its parts, it also divides the other, the number 9024 being equal to twice 3760 plus 1504. For the same reason, since 753 divides 1504, 753 will divide twice 1504. But 753 divides 3760; therefore 753 will divide 752, because 3760 is equal to twice 1504 plus 752, wherefore 753 will be contained one or more times in 752, which is absurd. Therefore 752 is the greatest common divisor of the two numbers 9024 and 3760.

In the same manner as above, we shall find that $\frac{195}{455}=\frac{3}{7}$; $\frac{580}{928}=\frac{5}{8}$; $\frac{714}{1785}=\frac{2}{5}$; $\frac{108}{684}=\frac{3}{19}$; $\frac{12989}{13827}=\frac{31}{33}$; $\frac{13549}{23113}=\frac{17}{29}$ and $\frac{698}{3343}=$ ———.

DIFFERENT WAYS OF CONSIDERING A FRACTION, AND CONSEQUENCES THENCE RESULTING.

96. The idea which we have hitherto given of a fraction is that the denominator shows of how many parts the unit is composed; and the numerator how many there are of these parts in the quantity expressed by the fraction.

We can yet consider a fraction in another point of view: we can consider the numerator as representing a certain quantity which ought to be divided into as many parts as there are units in the denominator. For example, in $\frac{4}{5}$, we may consider 4 as representing any four things whatsoever, 4 pounds, for example, which are required to be divided into five equal parts; for it is evident that it is the same thing to divide 4 pounds into 5 parts, in order to take one of these parts, or to divide 1 pound into 5 parts, in order to take 4 of these parts, and in general, *a fraction signifies the quotient of the numerator divided by the denominator.*

97. We can then consider the numerator of a fraction as a dividend, and the denominator as a divisor. From this we see what the remainders of the divisions signify in the form which we have given them. (60)

98. Hence it follows, 1st, that a whole number may always be expressed fractionally, in making this whole number the numerator, and giving it a unit for denominator; thus 8 is the same as $\frac{8}{1}$; 5 is the same as $\frac{5}{1}$, etc.

99. 2dly. That to convert any fraction into decimals, we have only to consider the numerator as the remainder of a division where the denominator was the divisor, and consequently to operate according to what has been said, (68, El. 2d.) observing first to put a cipher in the quotient, in the place of the units; thus we shall find that $\frac{3}{5}$ in decimals is 0,6; $\frac{5}{9}$ is 0,555 etc.; $\frac{1}{25}$ is 0,04, etc.

By this means we can reduce to decimals any proposed compound number. For example, if it be required to reduce 4 pounds, 16 shillings, 3 pence, 3 farthings, to the decimal of a pound, so as not to lose a half farthing, I observe that the pound contains 960 farthings, and consequently 1728 half farthings; we must then, not to lose a half farthing, carry the exactness of the operation beyond thousandths; that is to say as far as ten thousandths.

This established, I reduce 16*s*. $3\frac{3}{4}$*d*. all to farthings; and I have 783 farthings or $\frac{783}{960}$ of a pound, reducing this fraction to decimals, as has been shown, we have 0,8156 and consequently 4,8156*l*. for the proposed number.

100. We perform upon fractions the same operations as upon whole numbers.

OF THE ADDITION OF FRACTIONS.

101. If the given fractions have the same denominator, we add all the numerators, and give to the sum the common denominator of these fractions. Thus to add $\frac{2}{7}$, $\frac{3}{7}$, $\frac{5}{7}$, I add the numerators 2, 3, 5, and consequently, I have $\frac{10}{7}$, which I reduce to $1\frac{3}{7}$ (85.)

This is done upon the principle that if a number be divided by any other number, the quotient is equal to the sum of the

quotients when each of its parts are separately divided by that number.

102. If the fractions have not a common denominator, we commence by reducing them to a com. denom. according to what has been taught (90) and (91,) after which we add these new fractions as prescribed above. Thus, if we propose to add $\frac{3}{4}$, $\frac{2}{3}$, $\frac{4}{1}$, I change these fractions to these other three, $\frac{45}{60}$, $\frac{40}{60}$, $\frac{48}{60}$, of which the sum is $\frac{133}{60}$, which reduces to $2\frac{13}{60}$ (85.)

If we would find the sum of $5\frac{1}{3}$, $2\frac{7}{8}$, $3\frac{5}{6}$: I first say $5+2+3=10$, the sum of the whole numbers. Then (90) and

$$(91)\quad \frac{1}{3}+\frac{7}{8}+\frac{5}{6}=\frac{8}{24}+\frac{21}{24}+\frac{20}{24}=\frac{8+21+20}{24}=\frac{49}{24}=2\frac{1}{24}$$

the sum of the fractions. Lastly, $10+2\frac{1}{24}=12\frac{1}{24}$ the required sum. Or, (86) $5\frac{1}{3}=\frac{16}{3}$; $2\frac{7}{8}=\frac{23}{8}$; $3\frac{5}{6}=\frac{23}{6}$; then

$$\frac{16}{3}+\frac{23}{8}+\frac{23}{6}=\frac{128}{24}+\frac{69}{24}+\frac{92}{24}=\frac{128+69+92}{24}=\frac{289}{24}=12\frac{1}{24}.$$

EXAMPLES FOR PRACTICE.

1. $\frac{1}{2}+\frac{2}{3}+\frac{3}{7}=1\frac{25}{42}$ and $\frac{3}{4}+\frac{5}{9}+\frac{6}{7}=2\frac{41}{252}$.
2. $\frac{2}{5}+\frac{4}{7}+\frac{5}{11}=1\frac{241}{385}$ and $\frac{2}{3}+\frac{5}{6}+\frac{7}{8}=2\frac{3}{8}$.
3. $\frac{1}{6}+\frac{6}{7}+\frac{13}{14}+\frac{1}{21}=2$ and $5\frac{1}{2}+6\frac{3}{4}+7\frac{5}{8}=19\frac{7}{8}$
4. $2\frac{1}{2}+1\frac{1}{9}+3\frac{6}{7}+\frac{67}{126}=8$ and $101\frac{1}{2}+315\frac{3}{7}+\frac{19}{28}=417\frac{11}{84}$.
5. $\frac{1}{9}+\frac{3}{7}+\frac{5}{21}+\frac{16}{63}=\frac{8}{9}$ and $\frac{1}{11}+\frac{3}{8}+\frac{23}{55}=1$.

OF THE SUBTRACTION OF FRACTIONS.

103. If the two proposed fractions have the same denominator, we subtract the numerator of the one from the numerator of the other, and we give to the remainder the common denominator. If we take $\frac{5}{9}$ from $\frac{8}{9}$ the remainder is $\frac{3}{9}$, which reduces to $\frac{1}{3}$ (93.)

104. If from $9\frac{5}{8}$ we would subtract $4\frac{7}{8}$, as we cannot subtract $\frac{7}{8}$ from $\frac{5}{8}$, we borrow a unit from 9, which reduced to eighths and added to $\frac{5}{8}$, makes $\frac{13}{8}$, from which taking $\frac{7}{8}$ there remains $\frac{6}{8}$; after which taking 4 from 8 which remains after having borrowed, there remains in all $4\frac{6}{8}$ or $4\frac{3}{4}$.

Or, (86) $9\frac{5}{8}=\frac{77}{8}$; $4\frac{7}{8}=\frac{39}{8}$ then $\frac{77}{8}-\frac{39}{8}=\frac{38}{8}=4\frac{6}{8}$ or $4\frac{3}{4}$.

105. If the fractions have not the same denominator, we reduce them to a common denominator (90) and (91), after which

we perform the subtraction as hath been shown. Thus to take $\frac{2}{3}$ from $\frac{3}{4}$ I change these fractions to $\frac{9}{12}$ and $\frac{8}{12}$, and subtracting 8 from 9, there remains $\frac{1}{12}$. That is to say,

$$\frac{3}{4}-\frac{2}{3}=\frac{9}{12}-\frac{8}{12}=\frac{9-8}{12}=\frac{1}{12}$$

In the same manner we shall find that $\frac{4}{7}-\frac{3}{5}=\frac{1}{35}$; $\frac{5}{6}-\frac{1}{3}=\frac{1}{6}$; $\frac{17}{7}-\frac{3}{4}=\frac{47}{28}$. also $65\frac{5}{17}-19\frac{11}{12}=45\frac{77}{204}$.

OF THE MULTIPLICATION OF FRACTIONS.

106. *To multiply a fraction by another fraction, we must multiply the numerator of the one by the numerator of the other, and the denominator by the denominator.* For example, to multiply $\frac{2}{3}$ by $\frac{4}{5}$, we shall multiply 2 by 4, which will give 8 for the numerator; in like manner multiplying 3 by 5, we shall have 15 for the denominator, and consequently $\frac{8}{15}$ for the product.

To understand the reason of this rule, we must recollect that to multiply a number by another, is to take the multiplicand as many times as the multiplier contains a unit. Thus to multiply $\frac{2}{3}$ by $\frac{4}{5}$, is to take $\frac{4}{5}$ times the fraction $\frac{2}{3}$, or more exactly, it is to take four times the fifth of $\frac{2}{3}$: now, in multiplying the denominator 3 by 5, we change the thirds to fifteenths, that is to say, to parts which are five times less; and in multiplying the numerator 2 by 4, we take these new parts four times; we then take four times the fifth part of $\frac{2}{3}$: which is in effect multiplying $\frac{2}{3}$ by $\frac{4}{5}$.

107. If we had a whole number to multiply by a fraction, or a fraction by a whole number, we should write the whole number in the form of a fraction, in giving it a unit for denominator: for example, if I have 9 to multiply by $\frac{4}{7}$, this will be $\frac{9\times4}{1\times7}=\frac{36}{7}=5\frac{1}{7}$.

We see then that to multiply a fraction by a whole number, or a whole number by a fraction, is to multiply the numerator of the fraction by the whole number.

108. If there be whole numbers joined to fractions, we must, before we multiply, reduce each of these whole numbers to a fraction of the same kind as that which accompanies it. For example, if we have $12\frac{2}{5}$ to multiply by $9\frac{1}{11}$, I

change (86) the multiplicand, to $\frac{63}{5}$ and the multiplier to $\frac{39}{4}$, and I multiply $\frac{63}{5}$ by $\frac{39}{4}$ according to the above rule (106,) which gives me $\frac{2457}{20}=122\frac{17}{20}$.

We might also perform this same operation, by first multiplying the whole number and the fraction of the multiplicand by the whole number of the multiplier, and afterward by the fraction of the same multiplier, in the following manner:

	$12\frac{3}{5}$	
	$9\frac{3}{4}$	
Product of 12 by 9 -	108	
of $\frac{3}{5}$ by 9 - -	$5\frac{2}{5}$ -	$\frac{3}{20}$
of 12 by $\frac{3}{4}$ - -	9	
of $\frac{3}{5}$ by $\frac{3}{4}$ - - - - -		$\frac{9}{20}$
	122	$\frac{17}{20}$

But this operation is generally less simple than the first.

When we multiply several fractions, such as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, which have several of their numerators and denominators alike, we may suppress in each suite an equal number of these like numbers, thus:

$$\frac{2\times\dot{3}\times\dot{4}+\dot{5}}{3\times\dot{4}\times\dot{5}\times6}=\frac{2}{6}=\frac{1}{3}$$

This is done upon the principle that a fraction is not altered when both its terms are multiplied by the same number, and therefore the multiplication of each term by the product of $3\times4\times5$ would be superfluous.

Examples.

Suppose that we have $\frac{9}{13}$ to multiply by $\frac{26}{27}$.

$$\frac{\overset{2}{\dot{9}\times\dot{26}}}{\underset{3}{13\times27}}=\frac{2}{3}$$

Having indicated the multiplication by the sign into, as usual, I find that the numerator 9 of the first fraction will di-

vide the denominator 27 of the second; also, that 13, the denominator of the first, will divide 26, the numerator of the second. I therefore divide and have $\frac{2}{3}$ for the product. This is done upon the principle that the value of a fraction is not altered when both its terms are divided by the same number.

Examples for Practice.

1. $\frac{2}{3}\times 3\frac{7}{11}\times 2\frac{3}{4}\times\frac{13}{14}=6\frac{4}{21}$ and $\frac{6}{7}\times\frac{4}{6}\times 14=7\frac{1}{2}$.
2. $\frac{5}{6}\times\frac{2}{3}\times\frac{7}{11}\times 16\frac{1}{6}=3\frac{31}{396}$ and $\frac{9}{13}\times 2\frac{2}{9}\times\frac{1}{2}=1$.
3. $24\times\frac{2}{3}=16$; $36\times\frac{3}{4}=27$; $42\times\frac{5}{7}=30$ and $42\times\frac{6}{7}=36$.

Let us observe, that if we have 27 to multiply by $\frac{2}{3}$, the product will be the same whatever value of $\frac{2}{3}$ we multiply by.

Thus $\frac{\overset{9}{\cdot\cdot 27}}{1}\times\frac{2}{\underset{1}{\cdot 3}}=\frac{9\times 2}{1}=18$, now $\frac{2}{3}=\frac{6}{9}=\frac{18}{27}$, etc. and if we multiply by this last value of $\frac{2}{3}$ we shall have $\frac{\cdot\cdot 27}{1}\times\frac{18}{\cdot\cdot 27}=\frac{18}{1}$ $=18$ as before, and the same for any other value.

DIVISION OF FRACTIONS.

109. *To divide a fraction by a fraction, we must invert the two terms of the fraction which is the divisor, and multiply the fraction which is the dividend by this fraction thus inverted.*

For example, to divide $\frac{4}{5}$ by $\frac{2}{3}$, I invert the fraction $\frac{2}{3}$, which gives me $\frac{3}{2}$, I multiply $\frac{4}{5}$ by $\frac{3}{2}$ according to the rule given (106), and I have $\frac{12}{10}=\frac{6}{5}$ or $1\frac{1}{5}$ for the quotient of $\frac{4}{5}$ divided by $\frac{2}{3}$.

To perceive the reason of this rule, we must observe, that to divide $\frac{4}{5}$ by $\frac{2}{3}$ is to seek how many times $\frac{4}{5}$ contains $\frac{2}{3}$. Now it is easy to see that since the divisor is a third part of 2, it will be contained in the dividend three times as often as if it were 2; we must then first divide by 2, and afterward multiply by 3, which is nothing else than to take three times the half of the dividend, or to multiply by $\frac{3}{2}$, which is the divisor fraction inverted. In the same manner, we find that $\frac{13}{17}\div\frac{7}{9}=\frac{117}{119}$ and $\frac{27}{28}\div\frac{9}{14}=1\frac{1}{2}$.

110. If we had a fraction to divide by a whole number, or

a whole number to divide by a fraction, we should commence by putting the whole number under the form of a fraction, in giving it a unit for a denominator: for example, if we have 12 to divide by $\frac{5}{7}$, this is the same as to divide $\frac{12}{1}$ by $\frac{5}{7}$, which, according to the rule we have given, is to multiply $\frac{12}{1}$ by $\frac{7}{5}$, and this gives $\frac{84}{5}$ or $16\frac{4}{5}$. Likewise, if we had $\frac{3}{4}$ to divide by 5, we should operate thus: $\frac{3}{4} \div 5 = \frac{3}{4} \div \frac{5}{1} = \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$.

We see then that to divide a fraction by a whole number is to multiply the denominator by that number.

111. If there be whole numbers joined to fractions, we reduce these whole numbers each to a fraction of the same kind as that which accompanies it.

For example, if we have $54\frac{3}{5}$ to divide by $12\frac{2}{3}$, we change the dividend to $\frac{273}{5}$, and the divisor to $\frac{38}{3}$, we have then $\frac{273}{5}$ to divide by $\frac{38}{3}$, that is to say, (109) $\frac{273}{5}$ to multiply by $\frac{3}{38}$, which gives $\frac{819}{190}$ or $4\frac{59}{190}$.

SOME APPLICATIONS OF THE PRECEDING RULES.

112. From what we have said (96), it is easy to perceive how we can find the value of a fraction. If we demand, for example, the value of $\frac{5}{7}$ of a pound. Since $\frac{5}{7}$ of a pound is the same thing (96) as the seventh of five pounds, I reduce the 5 pounds to shillings (57), and dividing the 100 shillings which they give by 7, I find 14 shillings for the quotient, and 2 for the remainder; I reduce these 2 shillings to pence, and dividing 24 pence by 7, I have 3 pence $\frac{3}{7}$. Therefore, the value of $\frac{5}{7}$ of a pound is 14 shillings 3 pence and $\frac{3}{7}$ of a penny.

If we demand the value of $\frac{5}{7}$ of 24 pounds, it is plain that we could first take as above the $\frac{5}{7}$ of a pound, and then multiply the result by 24; but it is easier first to multiply $\frac{5}{7}$ by 24 pounds, which (107) gives $\frac{120}{7}$ pounds, and then to find the value of this fraction, which is 17*l.* 2*s.* $10\frac{2}{7}$*d.*

113. Decimal fractions, having no denominator, are yet easier to estimate. If we demand, for example, the worth of 0,532 of a fathom; as the fathom is 6 feet, I shall multiply 0,532 by 6, which will give me 3,192 feet; that is to say, 3 feet and 0,192 of a foot; multiplying this last fraction by 12. to have the value in inches, we shall have 2,304 inches, that

is to say, 2 inches and 0,304 of an inch; that is to say, the value of the fraction 0,532 of a fathom is 3 feet 2 inches and 0,304 of an inch.

Reciprocally, to convert the inferiour orders of a proposed compound number to decimal parts of the principal unit, we must commence with the units of the least kind, and divide successively by the number which signifies how many times one of these is contained in a unit of the next higher order.

Thus, in the preceding example, if I would bring back 0 fath. 3 ft. 2,304 in. to the decimal parts of a fathom, I divide 2,304 inches by 12, which gives me 0,192 of a foot. I then have in all 0 fath. 3,192 feet; and dividing these by 6, I have 0,532 of a fathom.

114. The estimation of fractions naturally leads us to speak of *compound fractions*. We style thus a succession of fractions, separated one from the other by the preposition of. For example, $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$, etc. are compound fractions. We reduce these to a single fraction, in multiplying all the numerators together and all the denominators together, the same as in multiplication; so that the fraction $\frac{2}{3}$ of $\frac{3}{4}$ is reduced to $\frac{6}{12}$ or $\frac{1}{2}$, the fraction $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$ is reduced to $\frac{30}{72}$, or $\frac{5}{12}$.

In effect, it is easy to see that to take $\frac{2}{3}$ of $\frac{3}{4}$ is nothing else than to multiply $\frac{3}{4}$ by $\frac{2}{3}$, since it is to take $\frac{2}{3}$ times the fraction $\frac{3}{4}$. In like manner, to take $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$, is to take $\frac{6}{12}$ of $\frac{5}{6}$, since $\frac{2}{3}$ of $\frac{3}{4}$ is equal to $\frac{6}{12}$; and what has been said shows that $\frac{6}{12}$ of $\frac{5}{6}$ becomes $\frac{30}{72}$, or $\frac{5}{12}$.

If we require the $\frac{3}{4}$ of $5\frac{3}{8}$, we convert the whole number 5 to eighths, and the question is then to find the value of $\frac{3}{4}$ of $\frac{43}{8}$, which is $\frac{129}{32}$, or $4\frac{1}{32}$.

Let us add to all that has been said upon fractions, an example which requires the use of several of the rules which we have established.

Suppose that we would construct a vessel $140\frac{2}{3}$ feet in length; that the distances between the ports, comprehending also the distance between the first port and the rabbit of the stem, and the space between the last port and the rabbit of the stern, make $108\frac{3}{4}$ feet: we demand whether we can pierce 12 ports in the first battery of each side.

From $140\frac{2}{3}$ feet I subtract $108\frac{3}{4}$ (103 and fol.;) there re-

mains $31\frac{11}{12}$ for the ports; I divide $31\frac{11}{12}$ by 12; that is to say, $\frac{383}{12}$ by $\frac{12}{1}$ (86) and (110), I have for the quotient $\frac{383}{144}$, which is equal to 2 feet and $\frac{95}{144}$, this fraction being equal to $7\frac{11}{12}$ inches, we must, therefore, give to each port 2 feet $7\frac{11}{12}$ inches, that is to say, 2 feet 8 inches nearly, which is a measure suitable for a vessel of $140\frac{2}{3}$ feet.

115. When a fraction, expressed by numbers somewhat considerable, is not reducible by the method given (95.) and we can be satisfied with having an approximate value, we obtain this by the following method, which alternately gives fractions greater and less than the proposed fraction, but always nearer and nearer, so that at the last operation we fall upon the proposed fraction. Let us take, for example, the fraction $\frac{100000}{314159}$, which, as we shall see in geometry, expresses the relation very nearly of the diameter of a circle to its circumference, and let us propose to express this fraction by other fractions less exact, it is true, but expressed by more simple numbers.

Divide the numerator and the denominator by the numerator; you will have $\frac{1}{3\frac{14159}{100000}}$; to have a first approximate value, neglect the fraction which accompanies 3, and you will have $\frac{1}{3}$ for the first approximate value, but this is rather too great.

To have a more approximate value, divide the numerator and the denominator of the fraction, which accompanies 3, each by the numerator of that fraction, and you will have $\frac{1}{3-\frac{1}{7\frac{887}{14159}}}$; neglect the fraction which accompanies 7, and you will have $\frac{1}{3\frac{1}{7}}$, or (86) $\frac{1}{\frac{22}{7}}$, or (103) $\frac{7}{22}$, for the second value, which is nearer than the first, but rather too little.

To have the value yet nearer, divide the numerator and the denominator of the fraction which accompanies 7, each by the numerator of that fraction, you will have $\frac{1}{3-\frac{1}{7-\frac{1}{15\frac{113}{887}}}}$;

suppress the fraction which accompanies 15. and you will have $\frac{1}{3\frac{1}{\frac{7}{17}}}$ which becomes $\frac{106}{355}$. value still nearer, but rather too great.

To have the value yet nearer, divide the two terms of the fraction which accompanies 15 each by the numerator 854, and you will have $\frac{1}{3\frac{1}{\frac{1}{15\frac{1}{1\frac{32}{171}}}}}$; neglecting the fraction $\frac{32}{171}$, you will have for the value still nearer $\frac{113}{355}$, but which is rather too little. We now see how the operation might be continued.

OF COMPOUND NUMBERS.

116. Although the preceding rules might also serve to calculate compound numbers, we think, however, that we should consider these in a more particular manner, because the division which we make of the principal unit often facilitates the calculation.

There are many kinds of compound numbers, and the rules for calculating them depend much upon the division which is made of the unit: however, it is not necessary to examine all these kinds in order to be able to calculate them; but it is of importance to know what relation their different parts have, as well to each other as with regard to the principal unit; for this reason, we here give the tables of the compound numbers in general use.

FEDERAL MONEY OF THE UNITED STATES OF AMERICA.

10 mills *m.* make - - -	1 cent, *c.*
10 cents - - -	1 dime, *d.*
10 dimes (or 100 cents) - -	1 dollar, $.
10 dollars - - -	1 eagle, *E.*

Note.—Accounts are kept in dollars and cents.

BRITISH MONEY.

4 farthings *qr.* make	- -	1 penny, *d.*
12 pence -	- -	1 shilling, *s.*
20 shillings -	- -	1 pound, *l.*

The farthings are written thus:

$\frac{1}{4}$ one farthing.
$\frac{1}{2}$ two farthings, or halfpenny.
$\frac{3}{4}$ three farthings.

Table showing the value of the American and Spanish Dollar in the currency of the different States, and British Dominions.

		s.	d.	
United States.	In New-Hampshire,	6	0	are a dollar.
	— Maine,			
	— Massachusetts,			
	— Rhode Island,			
	— Connecticut,			
	— Virginia,			
	— Kentucky,			
	— Tennessee,			
	— New-York,	8	0	
	— North Carolina,			
	— New-Jersey,	7	6	
	— Pennsylvania,			
	— Delaware,			
	— Maryland,			
	— South Carolina,	4	8	
	— Georgia,			
	— Canada,	5	0	
	— Nova Scotia,			
	— English, sterling,	4	6	
	— Irish,	4	10½	

TROY WEIGHT.

24 grains *gr.* make	-	1 pennyweight, *dwt.*
20 pennyweights -	-	1 ounce, *oz.*
12 ounces -	-	1 pound, *lb.*

Gold, Silver, and Jewels, are weighed by this weight.

APOTHECARIES WEIGHT.

20 grains *gr.* make - -	1 scruple, ℈
3 scruples - - -	1 drachm, ʒ
8 drachms - - -	1 ounce, ℥
12 ounces - - -	1 pound, ℔

Apothecaries mix their medicines by this weight, but buy and sell by Avoirdupois.

AVOIRDUPOIS WEIGHT.

16 drachms *dr.* make -	1 ounce, *oz.*
16 ounces - -	1 pound, *lb.*
28 pounds - -	1 quarter, *qr.*
4 quarters, or 112 pounds,	1 hundred weight, *cwt.*
20 hundred weight -	1 ton, *T.*

By this weight are weighed things of a coarse and drossy nature. Groceries, Chandlers' Wares, and all metals but gold and silver.

LONG MEASURE.

3 barley corns *b. c.* make -	1 inch, *in.*
12 inches - - - -	1 foot, *ft.*
3 feet - - - -	1 yard, *yd.*
$5\frac{1}{2}$ yards - - - -	1 rod or pole, *p.*
40 poles - - -	1 furlong, *fur.*
8 furlongs, or 1760 yards -	1 mile, *M.*
3 miles - - -	1 league, *L.*
60 Geographic, or $69\frac{1}{2}$ Statute miles	1 degree, *deg.*

360 degrees, the circumference of the earth.

Note. A fathom is 6 feet, and is used only to measure the depth of water.

A hand is 4 inches, and is used only to measure the height of horses.

The surveyor's, or four pole chain, is used to measure distances.

$7\frac{92}{100}$ inches make - -	1 link
100 links, or 66 feet - - -	1 chain
80 chains - - - -	1 mile.

Long Measure is used where length only is considered.

CLOTH MEASURE.

2¼ inches *in.* make -	1 nail, *n.*
4 nails - - -	1 quarter of a yard, *qr.*
2½ quarters, or 10 nails	1 ell Hamburgh, *E. H.*
3 quarters - -	1 ell Flemish, *E. F.*
4 quarters - -	1 yard, *yd.*
5 quarters - -	1 ell English, *E. E.*
6 quarters - -	1 ell French, *E. Fr.*

The name of this measure bespeaks its use.

LAND, OR SQUARE MEASURE.

144 square inches make -	1 square foot, *S. F.*
9 —— feet - - -	1 —— yard, *S. yd.*
30¼ —— yards - -	1 —— pole, *P.*
40 poles in length and one in breadth	1 rood, *R.*
4 roods - - -	1 acre, *A.*

The surveyor's chain is used to measure land: 10 square chains are one acre.

Note.—This measure respects length and breadth.

SOLID, OR CUBIC MEASURE.

1728 solid inches make - -	1 cubic foot, *C.F.*
27 cubic feet - - -	1 —— yard, *C.Y.*
40 feet round, and 50 feet square timber	1 ton, *T.*
Firewood, 8 feet long, 4 broad, and 4 high, or 128 solid feet -	1 cord, *C.*

By solid measure are measured such things as have length, breadth, and thickness.

LIQUID MEASURE.

2 pints *pts.* make - -	1 quart, *qt.*
4 quarts - - -	1 gallon, *gal.*
31½ gallons - - -	1 barrel, *bar.*
42 gallons - -	1 tierce, *tier.*
63 gallons - - -	1 hogshead, *hhd.*
84 gallons - -	1 puncheon, *pun.*
2 hogsheads, or 126 gallons -	1 pipe or butt, *p. b.*
2 pipes, or 252 gallons -	1 tun, *T.*

All Brandies, Spirits, Cider, Vinegar, Molasses, Oil, etc. are measured by this measure. Honey is sold by the pound Avoirdupois.

231 cubic inches make a gallon, and 10 gallons make an anker.

DRY MEASURE.

2 pints, *pts.*	1 quart, *qt.*
8 quarts	1 peck, *p.*
4 pecks	1 bushel, *bush.*
36 bushels	1 chaldron of coal, *ch.*

This measure is applied to dry articles, such as Corn, Fruit, Seed, Roots, Salt, Sand, Oysters, Coal, etc.

TIME.

60 seconds, *sec.* make	1 minute, *m.*
60 minutes	1 hour, *h.*
24 hours	1 day, *d.*
7 days	1 week, *w.*
4 weeks	1 month, *mo.*
13 months, 1 day, and 6 hours, or 365 days and 6 hours	1 common, or Julian year *y.*

The odd six hours make every fourth year (which is called Bissextile, or leap year) one day more, and this is added to the 2d month (February,) giving it 29 days. But agreeably to the most exact observations, 365 days, 5 hours, 48 minutes, and 57 seconds, make a solar year.

The year is also divided into 12 calendar months, which in number of days are unequal, as follows:—

	Names.	*Days.*
1st month,	January, hath	31
2d	February,	28
3d	March,	31
4th	April,	30
5th	May,	31
6th	June,	30
7th	July,	31

8th month,	August,	31
9th	September,	30
10th	October,	31
11th	November,	30
12th	December,	31

MOTION.

60 seconds, " make	1 prime minute,	′
60 minutes	1 degree	°
30 degrees	1 sign,	s

12 signs, or 360 degrees, make the whole circle of the Zodiac.

ADDITION OF COMPOUND NUMBERS.

117. To perform this operation, we write all the proposed numbers under each other, so that all the parts of the same kind may be found in the same vertical column; and having drawn a line under the whole, we commence the addition with the parts of the least kind: if their sum does not compose a unit of the next higher order, we write it under the units of its kind; if it contain parts enough to compose one or more units of the next higher order, we write under this column only the excess of an exact number of units of this second kind, and we retain these to add them to their equals, upon which we operate in the same manner.

EXAMPLE I.

	l.	*s.*	*d.*	
We propose to add	227	14	8	
	2549	18	5	
	184	11	11	
	17	10	7	
	2979	15	7	Sum.

The sum of the pence is 31, which contains 2 shillings and 7 pence; I place the 7 pence under the column of pence, and I retain the 2 shillings, which I add to the units of the shillings, the sum of which is 15 shillings, of this I set down the unit figure 5, and I retain the ten to add it to the tens;

the sum of these is 5; and as there must be two tens of shillings to make one pound, I take the half of 5 which is 2, with 1 for remainder; I set down this remainder, and I carry the two pounds to the column of pounds, which I add as usual.

EXAMPLE II.

	A.	*R.*	*P.*	*S.yds.*	*S.F.*
We propose to add	3	3	36	26	8
	36	2	27	19	7
	46	3	13	25	5
	16	3	35	21	6
	25	1	12	12	4
	129	3	6	15	

Here the sum of the square feet is 30, which contains 3 square yards, and there remains 3 feet; I set down 3 ft. and add 3 yards to the unit column of yards. I find that the sum of the yards is 106, and as $30\frac{1}{4}$ square yds. make a pole, I divide 106 by $30\frac{1}{4}$ or (86) by $\frac{121}{4}$, that is (106 and 109,) I multiply 106 by $\frac{4}{121}$, which gives me $\frac{424}{121}=3$ poles $\frac{61}{121}$.

Now it is evident that if this fraction of a pole be reduced to yards it will stand thus, $\frac{61}{121}\times\frac{121}{4}=\frac{61}{4}=15\frac{1}{4}$ yds., the sum total then of the yards is equal to 3 p. $15\frac{1}{4}$ yds., wherefore I have 15 yds. to write under yards, and 3 p. to carry to the unit column of the poles. But before I proceed to the addition of this column, I shall take notice of the $\frac{1}{4}$ of a yard which remained in dividing by 4. As it would be awkward to leave a fraction connected with the yards, while we have a lower denomination in the question, I reduce this fraction to feet in multiplying it by 9, which gives $2\frac{1}{4}$ ft. and adding this to the 3 feet which I first placed under the column of feet, I have $5\frac{1}{4}$ feet, which I leave under that column. I then proceed to the addition of the poles, and adding the 3 poles which I retain from the division of the yards, I find that the sum is 126, dividing by 40, because 40 poles make a rood, I have 3 roods 6 poles, I write 6 under poles, and add 3 to the roods, the sum of these is 15, dividing by 4, I have 3 acres, 3 roods. I write the 3 roods under roods, and add the 3 acres to

the unit column of acres. Having completed the addition, the sum total is 129 a., 3 r., 6 p., 15 sq.yds., $5\frac{1}{4}$ sq.f. (106.)

SUBTRACTION OF COMPOUND NUMBERS.

118. Write the proposed numbers under each other, as in addition, and commence the subtraction by the units of the lowest order. If the lower number can be subtracted from the upper one, write the remainder underneath. If it cannot be subtracted, borrow from the next higher order a unit, which you will reduce to the order upon which you are operating, and add to the number from which you cannot subtract. Do the same for each kind, and, when you have been obliged to borrow, add a unit to the next higher order of the lower number, which is the same thing as if you diminished by a unit the number from which you borrowed. Lastly, write each remainder, as it shall be found, under the number which gave it.

EXAMPLE I.

	l.	*s.*	*d.*
From -	143	17	6
we would take	75	12	9
	68	4	9

As 9*d.* cannot be subtracted from 6*d.* I borrow 1*s.*, which is 12 pence; this added to 6, gives 18, from which taking 9, there remains 9; I then add 1*s.* to 12, which makes 13, and subtracting this from 17, there remains 4: lastly I subtract 75*l.* from 143*l.*, and there remains 68*l.*

EXAMPLE II.

	A.	*R.*	*P.*	*S.yds.*	*S. F.*
From -	129	3	6	15	$5\frac{1}{4}$
we would take	125	3	9	18	$8\frac{1}{2}$
	3	3	36	26	8

Here, as I cannot subtract $\frac{1}{2}=\frac{2}{4}$ from $\frac{1}{4}$, I borrow a square foot, then (86) $1+\frac{1}{4}=\frac{5}{4}$ and $\frac{5}{4}-\frac{2}{4}=\frac{3}{4}$, which I write underneath. Having borrowed a unit from the 5 *S. F.*, instead of

diminishing this by a unit, I add a unit to the 8, which makes 9, then as I cannot take 9 from 5, I borrow a square yard, reducing this to square feet and adding it to the 5, I have 14, and taking 9 from 14 there remains 5, which I write under 8. Having borrowed a square yard, I add 1 to 18, which makes 19, and as this is greater than 15, I borrow 1 pole, which reduced to yards and added to 15, makes $45\frac{1}{4}$, subtracting 19, there remains $26\frac{1}{4}$; wherefore I write 26 underneath, and reducing $\frac{1}{4}$ of a square yard to square feet, I have $2\frac{1}{4}$ square feet; this added to $5\frac{3}{4}$ gives 8 square feet, which I leave under the column of feet. Having borrowed a pole, I add 1 to 9 which makes 10, then I say not 10 from 6, but borrowing a rood or 40 poles, I say 10 from 46 leaves 36, which I write underneath, and having borrowed a rood, I add 1 to 3, which makes 4, then borrowing an acre or 4 roods, 4 from 7 leaves 3, which I write under the roods: lastly I add 1 to 5, which makes 6, and 6 from 9 leaves 3, which being written underneath completes the operation.

This is a proof of the second example (117.) The number subtracted from, being the sum total of the five numbers added, the number subtracted, the sum of the four last, and the remainder the first.

3. Find the sum of 56A. 1R. 19P. 23yds. 8ft.....7A. 2R. 36P. 19yds. 3ft.....49A. 3R. 7P. 14yds. 7ft.....15A. 2R. 32P. 26yds. 8ft., and 233A. 1R. 29P. 12yds. 5ft.; after which prove by subtraction.

MULTIPLICATION OF COMPOUND NUMBERS.

119. We can always reduce the multiplication of compound numbers to the multiplication of a fraction by a fraction; the rule for which we have given (106.) For example, to find the cost of 54 sq. yds. 6 ft. of work, at the rate of 9*l.* 4*s.* 6*d.* per square yard; we reduce the multiplicand 9*l.* 4*s.* 6*d.* to pence (57,) which gives 2214 pence; and as the penny is the 240th part of a pound, the multiplicand may be represented by $\frac{2214}{240}$ of a pound; in like manner, we reduce the multiplier 54 sq. yds. 6 sq. ft. to feet, which gives 492 square feet; and as the square foot is the ninth part of a square yard, we have for the multiplier $\frac{492}{9}$ of a square

yard; so that the question is reduced to the multiplication of $\frac{3214}{240}$ by $\frac{492}{9}$, or (93) and (94) $\frac{369}{40}$ by $\frac{164}{3}$, which (106) gives $\frac{27716}{120}=\frac{5043}{10}$ pounds, or (112) 504*l.* 6*s.*

This method extends to all kinds of compound numbers; but as it generally requires more calculation than that which we are about to explain, we shall not dwell upon it any longer.

120. A number which is contained exactly in another number is called an aliquot part of that other: thus, 3 is an aliquot part of 12; the same may be said of 2, of 4, and of 6.

Let us recollect, that to multiply is nothing else than to take the multiplicand a certain number of times: to multiply by $8\frac{3}{4}$, for example, is to take the multiplicand 8 times, and to take it again $\frac{3}{4}$ of a time, or to take the $\frac{3}{4}$ of it. Now, we can take the $\frac{3}{4}$, either by taking the fourth and writing it three times, or by first taking the half, and afterward the half of this half: thus, to multiply 84 by $8\frac{3}{4}$, I write,

84	
8	
672	
42	half of 84.
21	half of 42, or one-fourth of 84.
735	product.

In multiplying 84 by 8, I have 672. Afterward, to take the $\frac{3}{4}$ of 84, I first take the half, which is 42; after which, for the remaining fourth, I take the half of 42, which is 21; and adding these three products, I have 735 for the total product.

121. To apply this to compound numbers, we must observe that the different kinds of units, inferiour to the principal unit, are fractions with regard to each other, and with regard to this principal unit; that, consequently, to multiply with facility these kinds of numbers, we must endeavour to divide them into aliquot parts of the principal unit, so that these aliquot parts may be employed conveniently; or to divide them into aliquot parts of each other; and if this division only furnishes aliquot parts which are inconvenient in

the calculation, we supply the defect by false produ
we shall exhibit in the following examples.

Example I.

We demand the cost of 54 sq. yds. $4\frac{1}{2}$ sq. ft. of the rate of $25 per square yard.

	$ cts.
We must multiply -	25 00
by - - - -	54yds $4\frac{1}{2}$ft
	100
	125
	12 50
	1362 50

We first multiply, in the usual manner, $25 by 54 which we multiply by $4\frac{1}{2}$ feet, which is half a sq[illegible] and which, consequently, ought to give only half [illegible] of a square yard, we take the half of $25, and [illegible] have $12 50 cents, for the true product.

Example II.

We have - -	[illegible]
to multiply by -	[illegible]
	[illegible]
	[illegible]
	[illegible]
	[illegible]
	[illegible]
	[illegible]

[illegible]

Note. In ordinary business, mills are not noticed unless they equal or exceed half a cent, in which case, we count one cent more.

EXAMPLE III.

If we have	$72		
to multiply by	5cwt.	3qr.	14lb.
	360		
	36		
	18		
	9		
	$423 product.		

Having multiplied by 5 *cwt.* we multiply by 3 *qr.* and to this end we divide this number into 2 *qr.* and 1 *qr.*; for 2 *qr.* we take the half of $72, which is $36; and for 1 *qr.* we take the half of 36 or the fourth of 72, which is 18. After which, to multiply by 14 *lbs.* instead of comparing these to the hundred weight, we compare them to the quarter, and as they are half a quarter, we take the half of 18, the price of a quarter, this gives $9. Lastly, having performed the addition, we have $423 for the product.

122. If the multiplicand be also a compound number, we proceed as in the following example:

EXAMPLE IV.

	l.	*s.*	*d.*
If we have	72	6	6
to multiply by	27cwt.	2qr.	21lb.
	504	0	0
	144		
	6	15	0
	1	7	0
	0	13	6
	36	3	3
	9	0	$9\frac{3}{4}$
	4	10	$4\frac{3}{4}+\frac{1}{2}$ *qr.*
	2002	9	$11\frac{1}{2}+\frac{1}{2}$ *qr.*

We first multiply 72*l.* by 27. Afterward, to multiply 6 shillings by 27, we divide these 6 shillings into 5*s.* and 1*s.* The 5 shillings being the fourth of a pound, ought, being multiplied by 27, to give 27 times the fourth of a pound or the fourth of 27*l.* which is 6*l.* 15*s.* To multiply 1*s.* by 27, we observe that a shilling is the fifth part of 5, by which we have multiplied; therefore we take the fifth of 6 pounds 15 shillings, which is 1*l.* 7*s.*

With regard to the 6 pence, as these are the half of a shilling, we take the half of 4*l.* 7*s.* which we had for a shilling.

So far the multiplicand is multiplied by 27. To multiply by 2 *qrs.* we take the half of 72*l.* 6*s.* 6*d.* which gives 36*l.* 3*s.* 3*d.*

Lastly, to multiply by 21*lb.* we divide this into 14*lb.* and 7*lb.* and as 14*lb.* is half a quarter, or the fourth of 2*qrs.* we take the fourth of 36*l.* 3*s.* 3*d.* which we had for 2*qrs.*; the quotient is 9*l.* 0*s.* $9\frac{3}{4}$*d.* also as 7*lb.* is the half of 14*lb.* we take the half of 9*l.* 0*s.* $9\frac{3}{4}$*d.* which gives 4*l.* 10*s.* $4\frac{3}{4}$*d.*$+\frac{1}{2}$*qr.* and adding all these parts, we have 2002*l.* 9*s.* $11\frac{1}{2}$*d.*$+\frac{1}{2}$*qr.* for the total product.

123. Thus far the parts of the multiplicand, which it has been necessary to take, have been very easy to estimate; but in the case when these parts would be complicate, we should operate as in the following example:

EXAMPLE V.

At the rate of 34*l.* 10*s.* 2*d.* per square yard, what is the price of 17*yds*?

238	0	0
34		
8	10	0
0	17	0
	2	10

586*l.* 12*s.* 10*d.*

Having multiplied 34*l.* by 17, and also the 10*s.* by 17, in taking the half of 17*l.* we multiply 2*d.* which is the sixth part of a shilling, and consequently the sixth part of the tenth part, or (114) the 60th part of 10 shillings; but instead of taking the 60th part of 8*l.* 10*s.* it is more convenient to make a false product, and first to take the tenth of the part given by 10 shillings, that is to say, the tenth of 8*l.* 10*s.*; this tenth, which is 17 shillings, is for 1 shilling; but as 2 pence is only the sixth of a shilling, we bar this false product, and write the sixth of it underneath. Having performed the addition, we have 586*l.* 12*s.* 10*d.* for the answer.

EXAMPLE VI.

How much work should we have for 34*l.* 10*s.* 2*d.* at the rate of 1*l.* for 17sq. yds.?

We must multiply 17 square yards by 34*l.* 10*s.* 2*d.*; that is to say, we must take 17 sq. yds. as often as the pound is contained in 34*l.* 10*s.* 2*d.*

sq. yds.	
17	
34*l.* 10*s.* 2*d.*	
$68^{\text{s. yds.}}$	$0^{\text{s. ft.}}$
51	
8	$4\frac{1}{2}$
0	$7\frac{13}{20}$
0	$1\frac{11}{40}$
$586^{\text{s. yds.}}$	$5\frac{31}{40}^{\text{ft.}}$

787159 A

Therefore, we first multiply 17 sq. yds. by 34; afterward, to multiply 17 yds. by 10*s*. we take the half of 17 sq. yds.; because 10*s*. is the half of a pound, and we have 8sq. yds. $4\frac{1}{2}$ sq. ft. To multiply by 2 pence, we seek, for greater facility, what a shilling would give, in taking the tenth of what 10*s*. gave; this tenth is 0 yds. $7\frac{13}{36}$ feet; we bar it, as it ought not to make a part of the product, but we take the sixth part of it to have the product of the 2 pence; we write this sixth, which is 0 sq. yds. $1\frac{11}{40}$ sq. ft. underneath, and, having added, we have 586 sq. yds. $5\frac{31}{40}$ ft. for the answer.

We have given this example principally to confirm what we have said, (45) viz. that it would be important to distinguish the multiplicand from the multiplier, when both are concrete. In effect, in the preceding example, as well as in this, the factors of the product are equally 17 sq. yds. and 34*l*. 10*s*. 2*d*.; yet the two products are different.

Examples for Practice.

1. What is the value of 216 bushels of potatoes, at 2*s*. 6*d*. a bushel? Ans. 27*l*.

2. What will 103 bushels of apples come to at 3*s*. 4*d*. a bushel? Ans. 17*l*. 3*s*. 4*d*.

3. What is the value of 177 yards of cloth at 6*s*. 8*d*. a yard? Ans. 59*l*.

4. How much will 369 bushels of wheat come to at 8*s*. 4*d*. a bushel? Ans. 153*l*. 15*s*.

5. How much will 20*T*. 15*cwt*. of hay amount to at $16 50 a ton? Ans. $342,37$\frac{1}{2}$.

6. What cost 30 yards of satin, at 6*s*. $11\frac{3}{4}$*d*. a yard? Ans. 10*l*. 9*s*. $4\frac{1}{2}$*d*.

7. What is the worth of 6374 yards of duck, at 3*s*. $6\frac{1}{4}$*d*. a yard? Ans. 1122*l*. 1*s*. $9\frac{1}{2}$*d*.

8. What is the value of a farm, containing 156*A*. 3*R*. 14*P*. at the rate of $26 25 an acre? Ans. $4116,984+.

9. How much does 14*cwt*. 3*qr*. 9*lb*. of sugar amount to at 2*l*. 14*s*. 7*d*. a cwt.? Ans. 40*l*. 9*s*. 6*d*. nearly.

DIVISION OF A COMPOUND NUMBER BY A SIMPLE NUMBER.

124. If the dividend alone be compound, and if, at the same time, the dividend and divisor consist of units of different kinds, we first divide the principal units of the dividend according to the ordinary rule: what remains after this division we reduce to units of the second order (57) these we add to those of the same kind which may be found in the dividend, and we divide the whole as usual: in like manner, we reduce the remainder of this division to units of the third kind, to which we add those of the same kind which may be found in the dividend, and we divide the whole as before: we continue to reduce the remainders to units of the next kind, as long as any inferiour kind is found in the dividend.

EXAMPLE.

We have given 4783*l.* 3*s.* 9*d.* for the payment of 87 sq. yds. of work; we demand how much this comes to per square yard.

l.	*s.*	*d.*	
4783	3	9	87
433			54*l.* 19*s.* 7*d.*
85			
1703*s.*			
833			
50			
609*d.*			
000			

We divide 4783*l.* 3*s.* 9*d.* by 87, in commencing with the pounds.

The 4783*l.* divided by 87, according to the ordinary rule, gives 54*l.* for the quotient, and 85 pounds for the remainder; these 85 pounds, reduced to shillings (57,) together with the 3 shillings of the dividend, give 1703 shillings, which, divided by 87, give 19 shillings for the quotient, and 50 shillings for the remainder; these 50 shillings, reduced to pence, give, with 9 pence of the dividend, 609 pence; which, divided by 87,

give, finally, 7 pence for the quotient. We have, therefore, 54*l.* 19*s.* 7*d.* for the answer.

125. But if the dividend and the divisor consist of units of the same kind, we must, before we make the division, examine whether the quotient should or should not be of the same kind, which the nature of the question always decides.

126. In the case where the dividend and the divisor being of the same kind, the quotient ought also to be of the same kind, the division is performed precisely as in the preceding case: for example, if we proposed this question—1243 pounds have produced a benefice of 7254 pounds, how much does this come to per pound? It is evident that the quotient should consist of units of the same kind with those of the dividend and the divisor; that is to say, should be pounds and that we should divide 7254 pounds by 1243; reducing the remainder of this division to shillings, as in the preceding example, and the second remainder to pence, we have 5*l.* 16*s.* $8\frac{760}{1243}$*d.* for the answer to the question.

127. But when the dividend and the divisor being of the same kind, the quotient should be of a different kind, we commence by reducing (57) the dividend and divisor each to the least kind of units mentioned in the dividend; after which we perform the division as in the preceding case, and we consider the units of the dividend as if they were of the same kind as those of which the quotient should consist: for example, if this question be proposed, how much work should be done for 7954*l.* 11*s.* 7*d.* at the rate of 72*l.* per square yard? It is evident from the nature of the question, that the quotient should be square yards and parts of a square yard; therefore, according to rule, we reduce 7594*l.* 11*s.* 7*d.* to pence, which gives 1909099*d.*; we also reduce 72*l.* to pence, which gives 17280*d.*; we then divide 1909099, considered as square yards, by 17280, and we have for the quotient 110 sq. yds. $4\frac{619}{1920}$ sq. ft. for the answer.

DIVISION OF A COMPOUND NUMBER BY A COMPOUND NUMBER.

128. When the divisor is also a compound number, we must reduce it to its lowest denomination (57), and multiply

the dividend by the number which shows how many there must be of the smallest units in the divisor to compose the principal unit of that same divisor; the division will then be reduced to the preceding case, where the divisor was simple.

Example.

If 53 cubic yards and 16 cubic feet of excavation cost 854*l.* 17*s.* 4*d.* what is the cost of one cubic yard ?

We must divide 854*l.* 17*s.* 4*d.* by 53 c. yd. 16 c. ft.; for this purpose, we reduce the 53 c. yd. 16 c. ft. to feet, which gives 1447 for the new divisor, and as there must be 27 feet to make a yard, which is the principal unit of the divisor, we multiply the proposed dividend 854*l.* 17*s.* 4*d.* by 27 (121), which gives 23081*l.* 8*s.* for the new dividend, so that we divide as follows :

```
23081l.  8s. | 1447
 8611        |-------
 1376        | 15l. 19s. 0 420/1447 d.
-------
27528s.
13058
   35
-------
  420d.
```

Dividing 23081*l.* by 1447, we have 15*l.* for the quotient, and 1376*l.* for the remainder. Then 1376*l.* reduced to shillings, together with the 8*s.* of the dividend, gives 27528*s.* which, divided by 1447, gives 19*s.* for the quotient and 35 shillings for the remainder. Reducing 35 shillings to pence, we have 420 pence; but as these do not contain the divisor 1447, we place 0 in the quotient, and signify the quotient of 420 divided by 1447 by placing these numbers in the form of a fraction as usual: so that the cost of one cubic yard is 15*l.* 19*s.* $0\frac{420}{1447}$*d.*

To understand the reason of this rule, we must observe that the 53 c. yd. 16 c. ft. are equal to 1447 cubic feet, and that the foot being the $\frac{1}{27}$ of a yard, the divisor is $\frac{1447}{27}$ cubic yards: now to divide by a fraction, we must (109) invert the divisor fraction, and multiply by this fraction thus inverted;

we must then multiply by $\frac{27}{1447}$, which requires that we should multiply by 27, and divide by 1447, as is also prescribed by the rule which we have given.

As the division of a compound number by a compound number is reduced, as we have seen, to the division of a compound number by a simple number, we should have the same regard to the nature of the units as in the articles (126) and (127.)

Examples for Practice.

1. If 216 bushels of potatoes cost 27*l.* what *is that* a bushel? Ans. 2*s.* 6*d.*

2. If 103 bushels of apples cost 17*l.* 3*s.* 4*d.* what is that a bushel? Ans. 3*s.* 4*d.*

3. Bought 177 yards of cloth for 59*l.* what is that a yard? Ans. 6*s.* 8*d.*

4. Bought 20 *T.* 15 *cwt.* of hay for $342,37½, what is the cost of a ton? Ans. $16,50.

5. Sold 6374 yards of duck for 1122*l.* 1*s.* 9½*d.* what was the price of a yard? Ans. 3*s.* 6¼*d.*

6. If a parcel of land containing 156 *A.* 3 *R.* 14 *P.* be worth $4116,984, what is the worth of one acre? Ans. $26,925 nearly.

OF THE FORMATION OF SQUARE NUMBERS, AND THE EXTRACTION OF THEIR ROOTS.

129. We call the *square* of a number the product which results from the multiplication of that number into itself; thus 25 is the square of 5, because 25 is the result of the multiplication of 5 by 5.

130. The *square root* of a proposed number is the number which, multiplied by itself, would reproduce that same proposed number: thus 5 is the square root of 25; 7 is the square root of 49.

$\sqrt{}$ *is the radical sign*, and when placed before a number signifies that the square root of it is to be extracted. Thus $\sqrt{49}=7$ is read the square root of forty-nine equal to seven.

131. A number which we square is then at the same time both multiplicand and multiplier; it is then twice factor (42) of

the product; for this reason we also call this product or square the *second power* of that number.

Note.—The number 2 is placed over a number to indicate its square or second power: thus 5^2 is the square or second power of 5. Hence 2 is called the index.

To square a number, no other art is required than to multiply it by itself, according to the ordinary rules of multiplication: but the extraction of the square root requires a method, at least, when the number or proposed square consists of more than two figures.

When the proposed number has only one or two figures, its root, in integers, is some one of the numbers

1, 2, 3, 4, 5, 6, 7, 8, 9,

the squares of which are

1, 4, 9, 16, 25, 36, 49, 64, 81.

The square root of 72, for example, is 8 for the whole number, because 72 being between 64 and 81, its root is between the roots of these numbers, that is to say, between 8 and 9, it is therefore 8 and a fraction, which fraction indeed we cannot exactly assign, but to which we can approach continually, as we shall see shortly.

132. The square root of a number which is not a perfect square, is called a *surd*, *irrational*, or *incommensurable* number.

133. Let us proceed to numbers which have more than two figures.

It is by observing what takes place in the formation of a square that we shall find the method which we should follow to obtain its root.

To square a number such as 54, for example:

$$\begin{array}{r} 54 \\ 54 \\ \hline 216 \\ 270 \\ \hline 2916 \end{array}$$

Having written the multiplicand and multiplier, as we see

here, we multiply, as usual, the 4 above by the 4 below, which evidently gives *the square of the units.*

We afterward multiply the 5 above by the 4 below, which gives *the product of the tens by the units.*

We then proceed to the second figure of the multiplier, and we multiply the 4 above by the 5 below, which gives the product of the units by the tens, or (44) *the product of the tens by the units.*

Lastly, we multiply the 5 above by the 5 below, which gives *the square of the tens.*

We add these products, and we have, for the square, the number 2916, which, as we see, is composed of *the square of the tens, plus twice the product of the tens by the units, plus the square of the units* of the number 54.

134. What we have observed being an immediate consequence of the rules of multiplication, is not more particularly applicable to the number 54 than to any other number composed of tens and of units; so that we can say universally that the square of any number composed of tens and units contains the three parts which we have signified, namely, the square of the tens of that number, twice the product of the tens by the units, and the square of the units.

135. This established, as the square of the tens is hundreds, (since $10 \times 10 = 100$) it is plain that the square of the tens cannot make a part of the two last figures of the total square.

In like manner the product of the double of the tens multiplied by the units, being consequently tens, cannot make a part of the last figure of the total square.

136. Therefore, from the square 2916, to return to its root, we can reason thus:

EXAMPLE I.

```
2916 | 54 root.
 416
 104
 ---
 000
```

Let us commence by finding the tens of this root: now the

formation of the square teaches us that there is in 2916 the square of these tens, and that this square cannot make a part of the two last figures: it is then in 29; and as the square root of 29 cannot be a whole number greater than 5, we thence conclude that the number of the tens of the root is 5, which we place by the side of 2916, as above.

We square 5, and subtract the product 25 from 29, there remains 4, to the side of which we bring down the two other figures 16 of the proposed number 2916.

To find the units of the root, let us observe what the remainder 416 contains; it only contains two parts of the square, namely, the double of the tens of the root, multiplied by the units, and the square of the units of this same root. Of these two parts, the first is sufficient to enable us to find the units which we seek: for since it is formed of the double of the tens multiplied by the units, if we divide it by the double of the tens which we know, it ought (74) to give the units for the quotient: the question then is, in what part of 416 is this double of the tens multiplied by the units contained? Now, we have remarked above that it cannot make a part of the last figure; it is therefore in 41; we must then divide 41 by the double (10) of the tens found; wherefore, we write the double (10) of the tens found under 41, and dividing, the quotient 4 that we find, is the number of the units, which we place on the right of the 5 tens already found, so that the required root is 54.

But we must observe that although the quotient 4 that we have found, be in effect the true quotient, the quotient found in this way, might sometimes be greater than it should be; because 41 (that is to say, the part which remains after the separation of the last figure) contains not only the double of the tens multiplied by the units, but also the tens arising from the square of the units; for this reason, to have no doubt with regard to the unit figure, we must employ the following verification:

Having found the figure 4 of the units, and written it in the root, we place it by the side of the double (10) of the tens, which gives 104, all the figures of which we multiply succes-

sively by the same number 4, and subtract the successive products from the corresponding parts of 416: as there is no remainder, we conclude that the root is in effect 54.

If any thing remained, the root would nevertheless be the true root in whole numbers, at least if this remainder were not greater than the double of the root, augmented by a unit: but this is not to be feared whilst we always take the greatest quotient.

The verification here taught, is founded upon the formation of the square itself; for, when we multiply 104 by 4, it is evident that we form the square of the units and the double of the tens multiplied by the units, that is to say, that which completes the perfect square.

137. From what has been said, we conclude that to extract the square root of a number which has not more than four figures, nor less than three, we must, after having separated two of them on the right, seek the square root of the period which remains on the left; this root is the number of the tens of the whole root sought, and we write it by the side of the proposed number, separating it by a line. We subtract from that same period the square of the root found; and, having written the remainder underneath this period, bring down on the right of this remainder the two figures which we had separated.

We separate by a point the unit figure of the period which we have brought down, and we divide the remaining figures on the left by the double of the tens, which we write underneath.

We write the quotient by the side of the first figure of the root, and also carry it to the side of the double of the tens which have served for a divisor.

Lastly, we multiply by this same quotient all the figures in this last line, and subtract their products in order as we find them from the corresponding figures above.

Let us complete the explanation of this by an example:

EXAMPLE II.

What is the square root of 7569?

```
75,69 | 87 root.
116,9
 167
 ----
 000
```

I separate the two figures 69, and seek the square root of 75; it is 8, which I write in the root. I square 8 and subtract from 75 the square 64; there remains 11, which I write under 75; I then bring down by the side of this same 11 the figures 69 which I had separated.

I separate in 1169 the last figure 9, to have in 116 the part which I ought to divide in order to find the units.

I form my divisor in doubling the 8 tens which I have found, and I write this divisor under 116, the division gives me for the quotient 7, which I write in the root on the right of 8.

I also carry this quotient to the side of 16; I multiply each figure of the number 167 which forms the last line by this same quotient 7, and subtract the products as fast as I find them from 1169; there remains nothing, which proves that 7569 is a perfect square, and the square of 87.

138. We must observe, that we ought to divide by the double of the tens, the part only which remains on the left after we have separated the last figure, so that if it does not contain the double of the tens, we do not on that account employ the figure separated; but we write 0 in the root. If, on the contrary, we find that the double of the tens is contained in it more than 9 times, we do not however place more than 9 in the root; the reason of this is the same as for division.(66)

139. Having well understood what we have said upon the square root of numbers which have not more than 4 figures, we shall easily comprehend what should be done when the number of figures is greater; seeing that of whatever number of figures the root may be composed, we may always con-

sider it as being composed of two parts, of which the one is tens and the other units; for example, 874 may be considered as representing 87 tens and 4 units.

This established, when we have found the two first figures of the root by the method which we have exposed, we can also find the third by the same method, in considering these two first figures as only making a single number of tens, and applying to them, in order to find the third, all that has been said of the first to find the second.

In like manner, when we have found the three first figures, if there should be a fourth, we consider the three first as only making a single number of tens, to which we apply, to find the fourth, the same reasoning that we applied to the two first to find the third, and so on.

But to proceed with order, we must commence by dividing the proposed number into periods of two figures each, going from the right hand to the left; the last on the left may contain only one.

The reason of this preparation is founded upon this; that considering the root as composed of tens and units, we must, according to what has been said above, (135 and fol.) commence by separating the two last figures on the right, to have, in the part which remains on the left, the square of the tens; but as this part is itself composed of more than two figures, a similar reasoning leads us again to separate two figures on the right, and so on.

Let us give an example of this operation.

EXAMPLE III.

What is the square root of 76807696?

```
76,80,76,96 | 8764 root.
128,0
  167
 ----------
  1117,6
   1746
 ----------
    7009,6
     17524
 ----------
    00000
```

Having separated the proposed number into periods of two figures each, going from the right hand to the left, I seek the square root of the left hand period 76: I find that it is 8, and I write 8 by the side of the proposed number: I square 8 and subtract the square 64 from 76; I have for the remainder 12, which I write under 76: to the right of this I bring down the period 80, the last figure of which I separate by a comma; and under the part 128, I write 16, the double of the root found: after which saying in 128 how many times 16? I find that it is contained seven times; I write 7 on the right of the root 8, and on the right of the double 16; I multiply 167 by this same number 7, and subtract from 1280 the product of this multiplication; there remains 111, to the side of which I bring down the period 76, which makes 11176; I separate the last figure 6 of this number, and under the part 1117 which remains on the left, I write 174, double of the root 87. I divide 1117 by 174, and having found 6 for the quotient, I write 6 in the root and by the side of 174: I multiply 1746 by this same number 6, and subtract 10476 from 11176, there remains 700; to the side of this remainder I bring down 96, the last figure of which I separate; under 7009, which remains on the left, I write 1752, double of the root found 876. In dividing 7009 by 1752, I find for the quotient 4, which I write in the root, and by the side of the double 1752: I multiply 17524 by this same number 4, and

subtracting the product from 70096 there remains nothing, therefore, the square root of 76807696 is exactly 8764.

140. When the proposed number is not a perfect square, there is a remainder at the end of the operation, and the square root found is the square root of the greatest square contained in the proposed number; in this case, it is not possible to extract the square root exactly; but we can approach to it as near as we think proper, that is to say, so that the error which would result in the square may be less than any given quantity.

This approximation is conveniently made by means of decimals. We must conceive on the right of the proposed number twice as many ciphers as we would have decimals in the root; perform the operation as usual, and afterward separate by a comma, upon the right of the root, half as many decimals as we have placed ciphers on the right of the proposed number. In effect(54), the product of the multiplication requiring as many decimals as there are in both the factors together, the square (of which the two factors are equal) ought to have the double of the decimals contained in one of the factors; that is to say, the double of those contained in the root.

Example IV.

We would have the square root of 87567 to less than a thousandth.

To make thousandths, there must be three decimals; we must then place six ciphers on the right of the square 87567: therefore, we must extract the square root of 87567000000.

```
8,75,67,00,00,00 | 295917
475
 49
 ----------
  346,7
   585
  ----------
   5420,0
    5909
   ----------
    10190,0
     59181
    ----------
     427190,0
      591827
     ----------
      129111
```

In performing the operation as in the preceding examples, we find for the square root, to within a unit, the number 295917; this root is that of 87567000000; but as we require that of 87567, or of 87567,000000, I separate half as many decimals in the root as I have added ciphers to the square, which gives me 295,917 for the square root of 87567 within a thousandth.

In like manner, if we require the square root of 2, to within a ten-thousandth, we shall extract the square root of 200000000 which we shall find to be 14142; separating the four figures on the right by a comma, we shall have 1,4142 for the square root of 2 to within a ten-thousandth.

141. We have seen(106) that to multiply a fraction by a fraction, we must multiply numerator by numerator, and denominator by denominator; consequently, to square a fraction, we must square the numerator and the denominator; thus the square of $\frac{2}{3}$ is $\frac{4}{9}$, that of $\frac{4}{5}$ is $\frac{16}{25}$.

142. Therefore, reciprocally to extract the square root of a fraction, we must extract the square root of the numerator and that of the denominator: thus the square root of $\frac{9}{16}$ is $\frac{3}{4}$, because that of 9 is 3, and that of 16 is 4.

143. But it may happen that the numerator or the denominator, or both, may not be perfect squares; if the numerator

only be not a perfect square, we shall extract the approximate root by the method we have given, and having extracted the root of the denominator, we shall place it as the denominator to the root of the numerator; thus if we would have the square root of $\frac{2}{9}$, we shall extract the approximate root of the numerator 2, which we shall find to be 1,4, or 1,41, or 1,414, or 1,4142, etc., according as we would have it more or less near; and as the square root of 9 is 3, we shall have for the approximate root of $\frac{2}{9}$, the quantity $\frac{1,4}{3}$, or $\frac{1,41}{3}$, or $\frac{1,414}{3}$, or $\frac{1,4142}{3}$, etc.

But if the denominator be not a square, we shall multiply the two terms of the fraction by this same denominator, which will not at all alter the value of the fraction, and will render the denominator a square; we shall then proceed as in the preceding case. For example, if we demand the square root of $\frac{3}{5}$, we shall change this fraction to $\frac{15}{25}$: in extracting the square root of 15 as far as three decimals, for example, we shall have 3,872; and as the square root of 25 is 5, the square root of $\frac{15}{25}$ will be $\frac{3,872}{5}$.

144. Not to have many kinds of fractions at a time, we shall reduce the result $\frac{3,872}{5}$, wholly to decimals, in dividing 3,872 by 5, which will give 0,774 for the root of $\frac{3}{5}$, expressed wholly in decimals.(99)

145. To conclude, if we have whole numbers joined to fractions, we reduce these whole numbers to fractions,(86) and operate as has been directed for a fraction. Thus, to extract the square root of $8\frac{3}{7}$, we change $8\frac{3}{7}$ to $\frac{59}{7}$, and this (143) to $\frac{413}{49}$, the approximate root of which we find to be $\frac{20,322}{7}$, or 2,903.

146. We may also reduce to decimals the fraction which accompanies the whole number; but we must be careful to employ an even number of decimals and double of those which we would have in the root; because the product of the multiplication of two numbers which have decimals, requiring to have as many decimals as there are in the two factors,(54) the square of a number which has decimals,

ought to have twice as many as that number. In applying this method to $8\frac{3}{7}$, we transform it to 8,428571,(99) the root of which is 2,903, as above.

147. In order to extract the square root of a decimal quantity, we should be careful to render the number of decimals even, if it be not so already; this is done in placing on the right of these decimals 1, 3, or 5, etc. ciphers; which does not alter the value of the given quantity.(30)

Thus, to extract the square root of 21,935, to within a thousandth, I extract the square root of 21,935000, which is 4,683; it is also that of 21,935. We shall also find that the square root of 0,542, to within a thousandth, is 0,736, and that the square root of 0,0054, to within a thousandth, is 0,073.

148. When we have found, by the method which has been explained, the three first figures of the root, we can have many others with greater facility and promptitude by division alone, in this manner:

Let us take, for example, 763703556823. I commence by seeking the three first figures of the root, according to the above method; I find 873 for this root, and 1574 for the remainder; I place by the side of this remainder the two figures 55, following the part 763703, which gave the three first figures, I should place the three following figures, if I had four figures in the root; four if I had five, and so on. I divide 157455, which I then have, by the double 1746 of the root; I find for the quotient 90; these are two new figures to place on the right of the root, which by this becomes 87390. I square this root, and I subtract its square 7637012100 from the part 7637035568, of which 87390 is the root; there remains 23468.

If I would have more figures in the root, as I have already five, I can, simply by division, find 4; I shall place, for this purpose, on the right of the remainder 23468, the two remaining figures 23 of the proposed number, and two ciphers, and dividing 234682300 by the double 174780 of the root found, I shall have 1342, for the four new figures which I place in the root; but in dividing the proposed number into periods, in the manner directed above, we see that its root

should only have six figures for the whole number; this root therefore is 373901.342, to within a thousandth.

We could, in general, carry the division one figure farther, that is to say, as far as the number of figures in the root already found; but there are some cases, though rare, where the error in the last figure would be five units; instead of which, in limiting ourselves to one figure less, as we have done, the last figure will never vary a unit.

If, after we have found the first figures of the root, by the ordinary method, the remainder be found equal to the double of the first figures, we must, to avoid all embarrassment, determine one more figure by the ordinary method, after which, we find the others by the above method, which, as we see plainly, applies also to decimals.

If any of the intermediate figures of the root should be ciphers, in the case where these ciphers should be among the number of the figures which we find by division, it might happen, if they ought to be the first figures of the quotient, that we should not perceive it, because in the division we do not mark the ciphers which ought to precede on the left of the quotient; we obtain these ciphers by observing that we ought always to have as many figures in the quotient as we have placed on the right of the remainder; and, consequently, when there are not enough, we must complete the number by ciphers placed on the left of this quotient.

The abridgment, we have here given, is a consequence of this general principle, which is easy to deduce from what we have seen,(134) namely, that the square of any quantity whatever, composed of two parts, contains the square of the first part, twice the first part multiplied by the second, and the square of the second.

Examples for Practice.

1. What is the square root of 16641? Ans. 129.
2. What is the square root of 549,9025? Ans. 23,45.
3. What is the square root of 8532769? Ans. 2921,09+.
4. What is the square root of 12674062,304225? Ans. 3560,065.

OF THE FORMATION OF CUBE NUMBERS, AND THE EXTRACTION OF THEIR ROOTS.

149. To form what is called the *cube* of a number, we must first multiply that number by itself, and afterward multiply by that same number the product resulting from this first multiplication.

Thus, the cube of a number is, properly speaking, the product of the square of the number multiplied by that same number: 27 is the cube of 3, because it results from the multiplication of 9, the square of 3, by the same number 3.

The number which we cube is then three times factor in the cube; it is for this reason that the cube is also called the *third power*, or third degree of this number.

150. Universally, we say that a number is raised to its second, third, fourth, fifth, etc. power, when we have multiplied it into itself 1, 2, 3, 4, etc. times successively, or when it is twice, 3 times, 4 times, 5 times, etc. factor in the product.

Note.—The numbers 2, 3, 4, 5, etc. are placed over a number to signify the second, third, fourth, fifth, etc. power of that number. Thus, $3^2=9$; $3^3=27$; $3^4=81$, etc.

151. The cube root of a proposed cube is the number, which, multiplied by its square, produces this cube; thus, 3 is the cube root of 27.

152. We have no need then of rules to form the cube of a number; but to return from the cube to its root we must have a method. We shall deduce this method from the examen of what takes place in the formation of a cube.

Let us observe, however, that we have no need of a method to extract the cube root in whole numbers, but when the proposed number has more than three figures; for 1000 being the cube of 10, every number under 1000, and, consequently, of less than four figures, will have for its root a number less than 10, that is to say, less than two figures.

Therefore, any number which shall fall between any two of these—

1, 8, 27, 64, 125, 216, 343, 512, 729,

will have its cube root, in integers, between the two corresponding numbers of this suite:

1, 2, 3, 4, 5, 6, 7, 8, 9,

of which the first contains the cubes.

153. Every number has not a cube root; but we can approach continually to a number which, being cubed, approaches also nearer and nearer to the proposed number; this we shall see after having learned to find the root of a perfect cube.

154. Let us see then of what parts the cube of a number is composed which contains tens and units.

Since the cube results from the square of a number multiplied by this same number, it is essential to recollect here (134) *that the square of a number composed of tens and units, contains* 1. *the square of the tens;* 2. *twice the product of the tens by the units;* 3. *the square of the units.*

To form the cube, we must then multiply these three parts by the tens and by the units of this same number.

To the end that we may perceive more distinctly the products which will thence result, let us give to this feigned operation the following form:

1st.

The square of the tens,	being multiplied by the tens, will give	The cube of the tens.
Twice the product of the tens by the units,		Twice the product of the square of the tens multiplied by the units.
The square of the units,		The product of the tens by the square of the units.

2d.

The square of the tens,	being multiplied by the units will give	The product of the square of the tens multiplied by the units.
Twice the product of the tens by the units,		Twice the product of the tens by the square of the units.
The square of the units,		The cube of the units.

Therefore, in collecting these six results and uniting those which are alike, we see that the cube of a number composed of tens and units, contains four parts, namely—*the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and lastly, the cube of the units.*

Let us form, from this, the cube of a number composed of tens and units, of 43, for example:

$$\begin{array}{r} 64000 \\ 14400 \\ 1080 \\ 27 \\ \hline 79507 \end{array}$$

We will then take the cube of 4, which is 64; but as this 4 is tens, its cube will be thousands; because $10^3 = 1000$; therefore the cube of the four tens will be 64000.

3×16, or three times the square of the 4 tens, being multiplied by the three units, will give 144 hundreds, because the square of 10 is 100; therefore the product will be 14400.

3 times 4, or three times the tens, being multiplied by the square 9 of the units, will give tens; hence this product will be 1080.

Lastly, the cube of the units will terminate in the place of units, and will be 27. Uniting these four parts, we shall have 79507 for the cube of 43, which cube we should without doubt have found more easily in multiplying 43 by 43, and the product 1849 again by 43; but it is not so much the question here to find the value of the cube, as to understand

by the examen of the parts which compose it, the method of returning to its root.

155. This established, see the process of the extraction of the cube root.

EXAMPLE I.

Let it be proposed then to extract the cube root of 79507.

Cube.	Root.
79,507	43
155,07	
48	

To have the part of this number which contains the cube of the tens of the root, I separate the three last figures of it, in which we have seen that this cube cannot be contained, since it is thousands.

I seek the cube root of 79; it is 4, which I write in the root. I cube 4, and subtract the product 64 from 79; there remains 15, which I write under 79.

To the side of 15 I bring down 507, which gives me 15507, in which there should be 3 times the square of the 4 tens found, multiplied by the units which we seek, plus 3 times these same tens multiplied by the square of the units, plus finally, the cube of the units.

I separate the two last figures 07; the part 155 which remains on the left contains 3 times the square of the tens multiplied by the units; for this reason, in order to have the units, (74) I shall divide this 155, by the triple square of the 4 tens, that is to say by 48. I find that 48 is contained three times in 155; I write therefore 3 in the root.

To prove this root, and find the remainder, if there be any, we could compose the three parts which should be found in 15507, and see if they form 15507, or by how much they differ from it; but it is as commodious to make this verification in directly cubing 43, that is to say, in multiplying 43 by 43, which produces 1849, and multiplying this product by 43, which gives finally 79507.

Therefore, 43 is exactly the cube root.

If the proposed number have more than six figures, we shall reason as in the following example:

EXAMPLE II.

Let it be proposed to extract the cube root of 596947688.

```
596,947,688 | 842
 849,47
 192
592704
-------------
  42436,88
  21168
596947688
-------------
000000000
```

We shall consider its root as composed of tens and units, and for this reason we shall commence by separating the three last figures.

The part 596947 which contains the cube of the tens, having more than three figures, its root will have more than one, and consequently it will have tens and units. We must then, to find the cube of these first tens, separate the three figures 947.

This done, I seek the root of 596; it is 8, which I write for the first figure of the root.

I cube 8, and subtract the product 512 from 596; there remains 84, which I write under 596.

To the side of 84 I bring down 947, which gives me 84947, the two last figures of which I separate.

Under the part 849, I write 192, which is the triple square of the root 8, and I divide 849 by 192; I find for the quotient 4, which I write in the root.

To verify this root, and to have at the same time the remainder, I cube 84 and subtract the product 592704 from the number 596947; I have for the remainder 4243.

To the side of this remainder I bring down the period 688, and considering the root 84 as a single number which signifies the tens of the root sought, I separate the two last figures 88 of the period brought down, and divide the part 42436 by the triple square of 84, that is to say, by 21168; the quotient is 2, which I write on the right of 84.

To verify the root 842, and to have the remainder, if there be any, I cube 842 and subtract the product 596947688 from the proposed number 596947688; and as there remains nothing, I conclude that 842 is the exact root of 596947688.

We must again observe, 1. that in the course of these operations we should never place more than 9 in the root.

2. If the figure placed in the root be too great, the subtraction cannot be performed; we must therefore diminish the root successively by 1, 2, 3, etc. units, till the subtraction becomes possible.

When the proposed number is not a perfect cube, the root which we find is only an approximate root, and it is seldom sufficient to have it in whole numbers. Hence decimals are again of great use in carrying the approximation as far as we please, without however enabling us ever to obtain an exact root.

156. To approach as near as we would to the cube root of an imperfect cube, we must place on the right of the proposed number three times as many ciphers as we would have decimals in the root, perform the extraction as in the preceding examples, and having finished the operation, we must separate by a comma, on the right of the root, as many figures as we proposed to have decimals.

Example III.

What is the cube root of 8755 to within a hundredth?

To have hundredths in the root, that is to say, two decimals, the cube or proposed number must have six;(54) we must therefore place six ciphers on the right of 8755.

The question then is to extract the cube root of 8755000000.

```
8,755,000,000 | 2061
-------------
07,55
12
8000
-------------
7550,00
1200
8741816
-------------
  131840,00
  127308
  8754552981
-------------
        447019
```

Pursuant to what has been said above, I separate this number into periods of three figures each, going from the right hand towards the left.

I extract the cube root of the last period 8; it is 2, which I write in the root. I cube 2 and subtract the product from 8; I have for the remainder 0, to the side of which I bring down the period 755, of which I separate the two last figures 55: under the remaining part 7, I write 12, triple square of the root, and dividing 7 by 12, I find 0 for the quotient, which I write in the root.

I cube the root 20; this gives me 8000, which being taken from 8755 leaves 755, to the side of which I bring down the period 000, of which I separate two figures on the right; under the remaining part 7550 I write 1200, triple square of the root 20; and dividing 7550 by 1200, the quotient is 6, which I write in the root.

I cube the root 206 and subtract the product from 8755000; I have for remainder 13184; to the side of this I bring down the last period 000, the two last figures of which I separate. Under the remaining part 131840, I write 127308, triple square of 206 the root found. I divide 131840 by 127308; the quotient is 1, which I write after 206; I cube 2061, and having subtracted from 8755000000, the product 8754552981, the remainder is 447019.

The approximate cube root of 8755000000 is then 2061; consequently that of 8755 is 20,61, since the cube has three times as many decimals as its root.(54)

If we would carry the approximation farther, we should place three ciphers on the right of the remainder, and continue the operation as we did each time that we brought down a period.

157. To multiply a fraction by a fraction, we must multiply numerator by numerator, and denominator by denominator; wherefore, in order to cube a fraction, we must cube the numerator and the denominator. Therefore, reciprocally, to extract the cube root of a fraction, we must extract the cube root of the numerator, and the cube root of the denominator. Thus, the cube root of $\frac{27}{64}$ is $\frac{3}{4}$, because the cube root of 27 is 3, and that of 64 is 4.

158. But if the denominator only be a cube, we shall extract the approximate root of its numerator, and give to this root for a denominator the cube root of the denominator. For example, if we demand the cube root of $\frac{143}{343}$; as the numerator is not a cube, I extract the approximate root, which is 5,22, to within a hundredth, and extracting the root of 343, which is 7, I have $\frac{5,22}{7}$ for the approximate root of $\frac{143}{343}$; or, in reducing it to decimals, (97) I have 0,74, for this approximate root, to within a hundredth.

159. If the denominator be not a cube, we multiply the two terms of the fraction by the square of this denominator, and then the new denominator being a cube, we proceed as has been said. For example, if we demand the cube root of $\frac{3}{7}$, I multiply the numerator and the denominator by 49, the square of the denominator 7; I have $\frac{147}{343}$, which (88) is of the same value as $\frac{3}{7}$. The cube root of $\frac{147}{343}$ is $\frac{5,27}{7}$; or, in reducing it entirely to decimals, 0,75, to within a hundredth.

If we had whole numbers joined to fractions, we should convert the whole to a fraction, and the question would then be reduced to the extracting of the cube root of a fraction, (157 and fol.)

Also, whether there be whole numbers or not, we can re-

duce the fraction to decimals; but we must be careful to carry this reduction to three times as many decimals as we would have in the root. Thus, if we would find the cube root of $7\frac{3}{11}$, to within a thousandth, we change the fraction $\frac{3}{11}$, to 0,272727272; so that to have the cube root of $7\frac{3}{11}$, we extract that of 7,272727272, which we find to be 1,937.

160. To extract the cube root of a number which has decimals, we must prepare it by a sufficient number of ciphers placed on the right, so that the number of its decimals may be 3, 6, or 9, etc.; we then extract the root as if there were no comma; and having performed the operation, we separate on the right of the root, by a comma, one-third of the number of decimals in the proposed quantity: and if the root have not a sufficient number of figures for the execution of this rule, we supply the defect by ciphers placed on the left of this root. Thus, to extract the cube root of 6,54, to within a thousandth, I place seven ciphers on the right, and extract the cube root of 6540000000, which is 1870; I separate three figures, because there are nine decimals in the cube, and I have 1,870, or simply 1,87, for the cube root of 6,54. We shall also find that the cube root of 0,0006, to within a hundredth, is 0,08.

161. When we have found the four first figures of the cube root by the method which we have explained, we can find others more promptly by division, and that in the following manner:

Suppose we demand the cube root of 5264627832723456; I seek the four first figures of it by the ordinary method; they are 1739, and the remainder of the operation is 5681413; by the side of this remainder, I place the two figures 72, which follow the part 5264627832, which gave these four first figures. I should place the three figures which follow this same part, if the root found had five figures, and four if it had six.

I divide 568141372 by 9072363, triple square of the root 1739; I have for the quotient 62, and these are two new figures to place on the right of 1739; so that 173962 is, in whole numbers, the cube root of the proposed number.

If we would carry it farther, we cube this root, and having

subtracted the product from the proposed number, we place on the right of the remainder four ciphers, and divide the whole by the triple square of 173962, which gives four decimals for the root.

We shall here make the same observation which we made (148) upon the case where the division does not give as many figures as it ought to give. And in these divisions, we will avail ourselves of the abridged rule given 69 and following.

* It is impossible to find exactly the square root of the numbers which are not the squares of the numbers 1, 2, 3, 4, 5, 6, etc.

Let us take, for example, the number 7. Since the root of this is greater than 2, and less than 3, this root, if we could find it, would be equal to 2 plus a fraction. Let us reduce the whole number to a fraction, let us add this fraction to the first, and reduce the fraction, which results from this addition to its most simple expression. Since the numerator and the denominator are prime one to the other, (that is to say, since one is not a multiple of the other,) the square of the numerator and the square of the denominator will be prime to one another.† Hence, the square of the denominator will never be a divisor of the square of the numerator: the quotient then of the division of the square of a numerator by the square of a denominator will never be a whole number; therefore, it is impossible that the square of this fraction should be equal to 7.

We might, in the same manner, demonstrate that it is impossible to find exactly the third, fourth, fifth, etc. roots, of the numbers which are not the third, fourth, fifth, etc. powers of the numbers 1, 2, 3, 4, 5, 6, etc.

† To prove the above position, that when two numbers are prime to one another, their squares are also prime to one another, let us take any two such numbers, for example 8 and 3; then, because 8 is not a multiple of 3, I say that the square of 8 is not a multiple of the square of 3, that is to say, that 64 is not a multiple of 9. For, if 64 be a multiple of 9, it will also be a multiple of 3, since 9 is the square of 3: now, if a number divide another and one of its parts, it will also divide the other part; but 3 divides 64, and it also divides 63, therefore, it divides 1, which is absurd. Therefore, 64 is not a multiple of 9.

Note.—The third, fourth, fifth, etc. roots are expressed by placing the numbers 3, 4, 5, etc. over the radical sign; thus $\sqrt[3]{27}$ signifies the cube root of 27; $\sqrt[4]{16}$ the fourth root of 16, etc.

TO EXTRACT THE FOURTH, FIFTH, SIXTH, ETC. ROOT OF A PROPOSED NUMBER.

Divide the proposed number into periods of four, five, or six figures, in going from the right hand towards the left; the root will have as many figures as there shall be periods.

This might be demonstrated by reasons similar to those which Bezout has given for the square or cube roots.

Let us take, for example, the number 116,2415,1851,9296. Divide this number into periods of four figures, going from the right hand towards the left; this number having four periods, its root will have four figures.

To have the figure of the thousands of the root, raise 2000 to the fourth power; if the fourth power of 2000 be greater than the proposed number, the figure of the thousands will be 1; if the fourth power of 2000 be less than the proposed number, raise 3000 to the fourth power; if the fourth power of 3000 be greater than the given number, the figure of the thousands will be 2; and if the fourth power of 3000 be less than the proposed number, raise 4000 to the fourth power; the fourth power of 4000 being greater than the proposed number, the figure 3 is the figure of the thousands.

To have the figure of the hundreds, raise successively 3100, 3200, 3300, etc. to the fourth power, until you find a power less and a power greater than the proposed number. But the fourth power of 3200 is less than the proposed number, and the fourth power of 3300 is greater than the proposed number. The figure of the hundreds then is 2.

If the fourth power of 3100 had been greater than the proposed number, it is evident that the figure of the hundreds would have been a cipher.

We should find in the same manner the figure of the tens and of the units.

If we would have decimals, we add as many times four ciphers as we would have decimals.

By similar reasonings, we should arrive at the extraction of the fifth, sixth, etc. root of a proposed number.

OF RATIOS, PROPORTIONS, AND PROGRESSIONS, AND OF SOME RULES DEPENDING UPON THEM.

162. The words *ratio* and *relation* have the same signification in mathematics, and both express the result of the comparison of two quantities.

163. If in the comparison of two quantities, we have in view to know by how much the one surpasses the other, or is surpassed by it, the result of the comparison, which is the difference of the two quantities, is called *arithmetical relation*.

Thus, if I compare 15 with 8, to find their difference 7, this number 7, which is the result of the comparison, is the arithmetical relation of 15 to 8.

To show that we compare two quantities in this manner, we separate the one from the other by a point; so that 15,8 shows that we consider the arithmetical relation of 15 to 8.

164. If in the comparison of two quantities, we purpose to know how many times the one contains the other, or is contained in it, the result of this comparison is called their *geometrical relation*. For example, if I compare 12 to 3 to know how many times 12 contains 3, the number 4, which expresses the number of times, is the geometrical ratio of 12 to 3.

To show that we compare two quantities in this sense, we separate the one from the other by two points; this expression 12 : 3 shows that we consider the geometrical relation of 12 to 3.

165. Of the two quantities that we compare, that which we read and write first is called *antecedent*, and the second is called *consequent*. Thus in the ratio 12 : 3, 12 is the antecedent, and 3 is the consequent: both are called *terms* of the ratio.

166. To have the arithmetical relation of two quantities, there is then nothing else to do than to subtract the less from the greater.

167. And to have the geometrical ratio of two quantities, we must divide the one by the other.

168. We shall henceforth estimate this ratio in dividing the antecedent by the consequent: thus the ratio of 12 to 3 is 4, and the ratio of 3 to 12 is $\frac{3}{12}$ or $\frac{1}{4}$.

169. An arithmetical relation is not changed when we add to each of its two terms, or when we subtract from each the same quantity, because the difference in which consists the relation is always the same.

170. A geometrical ratio is not changed when we multiply or when we divide each of its two terms by the same number; for the geometrical ratio consisting (168) in the quotient of the division of the antecedent by the consequent, is a fractional quantity which (88) cannot alter by the multiplication or the division of its terms by the same number. Thus the ratio of 3 : 12 is the same as that of 6 : 24, which we have in multiplying the two terms of the first by 2; it is the same as that of 1 : 4, which we have in dividing each by 3.

171. This property serves to simplify the ratios. For example, if I had to examine the relation of $6\frac{3}{4}$ to $10\frac{2}{3}$, I should say, in reducing all to fractions, this ratio is the same as that of $\frac{27}{4}$ to $\frac{32}{3}$, or in reducing to a common denominator, the same as that of $\frac{81}{12}$ to $\frac{128}{12}$, or finally suppressing the denominator 12, which is the same as to multiply the two terms of the ratio by 12, this ratio is the same as that of 81 to 128.

172. When four quantities are such that the ratio of the two first is the same as the ratio of the two last, we say that these four quantities form a *proportion;* and this proportion is arithmetical or geometrical, according as the ratio which we there consider is arithmetical or geometrical.

The four quantities 7, 9, 12, 14, form an arithmetical proportion, because the difference of the two first is the same as that of the two last. To signify that they are in arithmetical proportion, we write them thus, 7 . 9 : 12 . 14, that is to say, we separate by a point the two terms of each ratio, and the two ratios by two points. The point which separates the two terms of each ratio, signifies *is to;* and the two points which separate the two ratios, signify *as;* so

that to read the proportion thus written, we say, 7 *is to* 9; *as* 12 *is to* 14.

The four quantities 3, 15, 4, 20, form a geometrical proportion, because 3 is contained in 15, as 4 is contained in 20.

To signify that they are in geometrical proportion, we write them thus, 3 : 15 : : 4 : 20, that is to say, we separate the two terms of each ratio by two points, and the ratios by four points. The two points signify *is to;* and the four points signify *as;* so that we say 3 *is to* 15, *as* 4 *is to* 20.

It must be observed simply that, in the arithmetical proportion, we precede the word *as* with the word *arithmetically.*

173. The first and last terms of the proportion are called the *extremes;* the 2d and 3d are called the *means.*

As there are two ratios, and consequently two antecedents and two consequents, we say, for the first ratio, *first antecedent, first consequent;* and for the second, *second antecedent, second consequent.*

174. When the two mean terms of a proportion are equal, the proportion is called *continued:* 3 . 7 : 7 . 11 forms a continued arithmetical proportion: we write it thus, $\div$3 . 7 . 11; the two points and the bar which precede, are to show that in reading we should repeat the mean term which is here 7.

The proportion 5 : 20 : : 20 : 80 is a continued geometrical proportion, which by abbreviation we write thus: —$\div\div$5 : 20 : 80; the use of the four points and the bar is the same as in the continued arithmetical proportion.

175. It follows from what has been said upon arithmetical and geometrical proportions:

1. That if in an arithmetical proportion we add to each of the antecedents, or if we subtract from each the difference or ratio which reigns in this proportion, according as the antecedent shall be less or greater than its consequent, each antecedent will become equal to its consequent; for this is to give to the least term of each ratio that which it lacks of being equal to its neighbour; or to subtract from the greatest term of each that by which it surpasses its

neighbour. Thus in the proportion 3 . 7 : 8 . 12, add the difference 4 to the first and to the third term, and you will have 7 . 7 : 12 . 12, and it is easy to perceive that this is general.

2. If in a geometrical proportion you multiply each of the two consequents by the ratio, you will in like manner render each equal to its antecedent: for, to multiply the consequent by the ratio, is to take it as many times as it is contained in the antecedent: thus in the proportion 12 : 3 : : 20 : 5, multiply 3 and 5 each by 4, and you will have 12 : 12 : : 20 : 20 ; in like manner, in the proportion 15 : 9 : : 45 : 27, multiply 9 and 27 each by $\frac{15}{9}$ or $\frac{5}{3}$, which is the ratio, and you will have 15 : 15 : : 45 : 45.

PROPERTIES OF ARITHMETICAL PROPORTION.

176. The fundamental property of arithmetical proportion, is that *the sum of the extremes is equal to the sum of the means*: for example, in this proportion 3 . 7 : 8 . 12, the sum 3 and 12 of the extremes, and that 7 and 8 of the means are equally 15.

We can assure ourselves that this property is general, as follows:

If the two first terms were equal to one another, and the two last terms also equal to one another, as in this proportion: 7 . 7 : 12 . 12,
it is evident that the sum of the extremes would be equal to that of the means.

Now, every arithmetical proportion may be brought to this state,(175) in adding to the antecedent, or in taking from it the difference which reigns in the proportion. This addition, which will equally augment the sum of the extremes and that of the means, cannot in the least alter the equality of the two sums; therefore, if they become equal by this addition, it is because they were equal without this same addition. The reasoning is the same for the case of the subtraction.

177. Since the two mean terms in the continued proportion are equal, it follows from what we have shown, that, in this same proportion, the sum of the extremes is double of the mean term, or that the mean term is half the sum of the

extremes. Thus, to have an arithmetical mean between 7 and 15, I add 7 to 15; and taking the half of the sum 22, I have 11 for the mean term, so that ÷7 . 11 . 15.

* In every arithmetical proportion, the sum of the extremes is equal to the sum of the means.

Let us take, for example, the arithmetical proportion 5 . 3 : 6 . 4.

Since the difference between the first term and the second is the same as the difference between the third and the fourth, it is evident that we shall have,

$5-3=6-4$, in adding $3+4$ to these two equalities, we shall have $5-3+3+4=6-4+3+4$, that is to say, $5+4=6+3$.

Therefore, the sum of the extremes of this proportion is equal to the sum of the means.

If we had 3 . 5 : 4 . 6, we should have 5 minus 3, equal to 6 minus 4, and we should demonstrate in the same manner that 3 plus 6 is equal to 5 plus 4.

The reasoning would be the same for every other arithmetical proportion. Therefore, in every arithmetical proportion, the sum of the extremes is equal to the sum of the means.

If the arithmetical proportion were continued, it is evident that the sum of the extremes would be double of the mean term.

Four numbers are in arithmetical proportion, when the sum of the extremes is equal to the sum of the means.

Let us take, for example, the four numbers 5, 3, 6, 4, of which the sum of the extremes is equal to the sum of the means.

Since $5+4=6+3$, we shall have, in subtracting $4+3$ from these two equalities, $5+4-4-3=6+3-4-3$; that is to say, $5-3=6-4$.

The four numbers 5, 3, 6, 4, are then in arithmetical proportion, since the difference between the two first is equal to the difference between the two last.

Therefore, 5 . 3 : 6 . 4.

Four numbers are not in arithmetical proportion when the sum of the extremes is not equal to the sum of the means.

Let them be, for example, the four numbers 5, 3, 7, 4, of

which the sum of the extremes is not equal to the sum of the means.

Since $5+4$ is not equal to $7+3$, in subtracting $4+3$ from these two inequalities, $5+4-4-3$ will not equal $7+3-4-3$; that is to say, that $5-3$ will not equal $7-4$.

Therefore, the four numbers 5, 3, 7, 4, are not in arithmetical proportion, since the difference between the two first is not equal to the difference between the two last.

It is evident from the preceding, that if four numbers be in arithmetical proportion, these four numbers will also be in proportion, if we change the places of the extremes and the places of the means, and if we put the extremes in the place of the means and the means in the place of the extremes; for, in all these mutations, the sum of the extremes will always equal the sum of the means.

PROPERTIES OF GEOMETRICAL PROPORTION.

178. The fundamental property of the geometrical proportion is that *the product of the extremes is equal to the product of the means*; for example, in this proportion 3 : 15 :: 7 : 35, the product of 35 by 3, and that of 15 by 7, are equally 105.

We can satisfy ourselves that this property belongs to every geometrical proportion, as follows:

If the antecedents were equal to the consequents, as in this proportion,

$$3 : 3 :: 7 : 7$$

it is evident that the product of the extremes would be equal to the product of the means.

But we can always bring a proportion to this state (175) in multiplying the two consequents by the ratio. This multiplication will, it is true, make the product of the extremes a certain number of times greater than it would have been, or a certain number of times less, if the ratio be a fraction; but it will produce the same effect upon that of the means; therefore, since after this multiplication the product of the extremes would be equal to the product of the means, these two products should be equal without this same multiplication.

We can then take the product of the extremes for that the means, and reciprocally.

Therefore, *in the continued proportion, the product of the extremes is equal to the square of the mean term;* for the two means being equal, their product is the square of one of them. Therefore, to have a geometrical mean between two proposed numbers, we must multiply these two numbers together, and extract the square root of this product. Thus, to have a geometrical mean between 4 and 9, I multiply 4 by 9, and the square root 6 of the product 36 is the mean proportional sought.

* In every geometrical proportion, the product of the extremes is equal to the product of the means.

Let us take, for example, the geometrical proportion $12 : 3 :: 16 : 4$; I say that $12 \times 4 = 3 \times 16$.

In effect, since $12 : 3 :: 16 : 4$, we shall have $\frac{12}{3} = \frac{16}{4}$.

Reducing these two fractions to the same denominator, we shall have $\frac{12 \times 4}{3 \times 4} = \frac{16 \times 3}{4 \times 3}$; that is to say, $12 \times 4 = 16 \times 3$. But 12×4 is the product of the extremes, and 16×3 is the product of the means; therefore, in the proportion $12 : 3 :: 16 : 4$ the product of the extremes is equal to the product of the means.

If the proportion were continued, it is evident that the product of the extremes would be equal to the square of the mean term.

Four numbers are in geometrical proportion when the product of the extremes is equal to the product of the means.

In effect, since $12 \times 4 = 16 \times 3$, dividing these two equalities by 4×3, we shall have $\frac{12 \times 4}{4 \times 3} = \frac{16 \times 3}{4 \times 3}$; that is to say $\frac{12}{3} = \frac{16}{4}$; therefore, the four numbers 12, 3, 16, 4, are in geometrical proportion, because the quotient of the first divided by the second is equal to the quotient of the third by the fourth. The reasoning would be the same for any other numbers.

Four numbers are not in geometrical proportion when the product of the extremes is not equal to the product of the

Let the numbers be, for example, the four numbers 12, 3, 20, 4, of which the product of the extremes is not equal to the product of the means.

In effect, since 12×4 is not equal to 20×3, if we divide these two inequalities by 4×3, $\frac{12 \times 4}{4 \times 3}$ will not be equal to $\frac{20 \times 3}{4 \times 3}$; that is to say, that $\frac{12}{3}$ will not equal $\frac{20}{4}$:

The quotient then of the first by the second is not equal to the quotient of the third by the fourth; therefore, the numbers 12, 3, 20, 4, do not form a geometrical proportion. The reasoning would be the same for any other numbers.

From the preceding, if four numbers be in geometrical proportion, it is evident that if we change the place of the extremes and the place of the means, or if we put the extremes in the place of the means, and the means in the place of the extremes, these four numbers will still be in proportion; for, in all these mutations, the product of the extremes will be equal to the product of the means.

179. From the fundamental property of the geometrical proportion, it follows that, if knowing the three first terms of a proportion, we would determine the fourth, we must *multiply the second and third, and divide the product by the first;* for it is evident (74) that we should have the fourth term in dividing the product of the two extremes by the first term; now this product is the same as that of the means; therefore, we shall also have the fourth term in dividing the product of the means by the first term.

Thus, if we would have the fourth term of a proportion of which the three first are 3 : 8 : : 12 : . . . I multiply 8 by 12, which gives 96; this I divide by 3, and the quotient 32 is the fourth term sought, so that 3, 8, 12, 32, form a proportion; in effect, the first ratio is $\frac{3}{8}$, and the second $\frac{12}{32}$, which, (89) in dividing the two terms by 4, is also $\frac{3}{8}$.

By a similar reasoning, we see that we could find any other term of the proportion when we know three of them.

If the term which we would find be one of the extremes, we must multiply the two means, and divide by the known extreme:

if, on the contrary, we would find one of the means, we must multiply the two extremes, and divide by the known mean.

180. This property of the equality between the product of the extremes and that of the means, cannot appertain to four quantities which are not in geometrical proportion. In effect, if we have four quantities which are not in geometrical proportion, in multiplying the consequents by the ratio of the two first, there will only be the first antecedent, which will become equal to its consequent. For example, if we have 3, 12, 5, 10, in multiplying the consequents 12 and 10 by the ratio $\frac{1}{4}$ of the two first terms 3 and 12, we shall have 3, 3, 5, $\frac{10}{4}$ in which it is evident that the product of the extremes is not equal to that of the means; therefore, these products could not any more be equal, even though we had not multiplied the consequents by the ratio $\frac{1}{4}$, and it is plain that this reasoning is applicable to every case.

Therefore, *if four quantities are such that the product of the extremes is equal to the product of the means, these quantities are in proportion.* Hence we shall deduce this second property of proportions.

181. *If four quantities be in proportion, they will also be in proportion if we put the extremes in the place of the means, and the means in the place of the extremes.*

182. *The same will be the case, that is to say, the proportion will subsist if we change the places of the extremes or those of the means.*

In effect, in every case, it is easy to see that the product of the extremes will be equal to that of the means.

Thus the proportion 3 : 8 : : 12 : 32 furnishes all the following proportions by the simple permutation of its terms.

3 : 8 : : 12 : 32
3 : 12 : : 8 : 32
32 : 12 : : 8 : 3
32 : 8 : : 12 : 3
8 : 3 : : 32 : 12
8 : 32 : : 3 : 12
12 : 3 : : 32 : 8
12 : 32 : : 3 : 8

And it is the same in every other proportion.

183. Since we can put the third term in the place of the second, and reciprocally, we thence conclude *that we can, without disturbing a proportion, multiply or divide the two antecedents by the same number, and that it is the same with regard to the consequents;* for in making this permutation, the two antecedents of the given proportion will form the first ratio; and the two consequents the second. Therefore, to multiply the two antecedents of the first proportion becomes the same as to multiply the two terms of a ratio each by the same number, which (170) does not change this ratio. For example, if I have the proportion 3 : 7 : : 12 : 28, I can, in dividing the two antecedents by 3, say 1 : 7 : : 4 : 28, because from the proportion 3 : 7 : : 12 : 28 we conclude (182) that 3 : 12 : : 7 : 28; and in dividing the two terms of the first ratio by 3, we have 1 : 4 : : 7 : 28, which (182) may be changed to 1 : 7 : : 4 : 28.

184. *Every change made in a proportion, so that the sum of the antecedent and consequent, or their difference, may be compared to the antecedent or to the consequent, in the same manner in each ratio, will always form a proportion.*

For example, if we have the proportion

$$12 : 3 : : 32 : 8,$$

we can thence deduce the following proportions:

$$12+3 : 3 : : 32+8 : 8$$
$$12-3 : 3 : : 32-8 : 8$$
$$12+3 : 12 : : 32+8 : 32$$
$$12-3 : 12 : : 32-8 : 32$$

For if it be to the consequent that we compare, it is easy to see that the antecedent, augmented or diminished by the consequent, will contain this consequent a time more or a time less than before; and as this comparison is made in the same manner for the second ratio, which, by the nature of the proportion, is equal to the first, it follows that the two new ratios will also be equal to one another.

If it be to the antecedent that we compare, the same reasoning will still apply, in conceiving that in the proportion in which we make this change, we have put the antecedent of

each term in the place of its consequent, and the consequent in the place of the antecedent, which may be done. (181)

183. Since, in putting the first term of a proportion in the place of the second, and reciprocally, there is still a proportion, (182) we shall conclude that the two antecedents contain each other as many times as the consequents also contain one another.

Therefore, the sum of the two antecedents of every proportion contains the sum of the two consequents, or is contained in it as many times as one of the antecedents contains its consequent, or is contained in it.

For example, in the proportion

$$12 : 3 :: 32 : 8$$

$12+32 : 3+8 :: 32 : 8$, which is evident.

But to satisfy ourselves generally, we have only to observe, that if the first antecedent contain the second four times, for example, the sum of the two antecedents will contain the second five times, and, for the same reason, the sum of the consequents will contain the second consequent five times; therefore, the sum of the antecedents will contain that of the consequents as the quintuple of one of the antecedents contains the quintuple of its consequent, that is to say, (170) as one of the antecedents contains its consequent.

We might, in the same manner, prove that the difference of the antecedents is to the difference of the consequents, as one antecedent is to its consequent.

186. It is evident that the proposition which we have demonstrated leads to this: that if we have two equal ratios, for example,

That of - -	4 : 12
and that of - -	7 : 21
	11 : 33

We shall still have the same ratio, in adding antecedent to antecedent, and consequent to consequent.

Therefore, *if we have several equal ratios, the sum of all the antecedents is to the sum of all the consequents as any one of the antecedents is to its consequent.* For example, if we

have the equal ratios 4 : 12 : : 7 : 21 : : 2 : 6, we can say, that 4+7+2 : 12+21+6 : : 4 : 12 or 7 : 21, etc.

For, after having added the antecedents and the consequents of the two first ratios together, the new ratio which, as we have seen, will be the same as each of the two first, will also be the same as the third: consequently, we can add it to this, and we shall still have the same ratio, and so on for any number of ratios.

187. We call *compound ratio* that which results from two or more ratios, the antecedents of which we multiply together, and the consequents together. For example, if we have the two ratios 12 : 4 and 25 : 5, the product of the antecedents 12 and 25 will be 300; that of the consequents 4 and 5 will be 20; the ratio of 300 to 20 is what we call the ratio composed of the ratios of 12 to 4 and of 25 to 5.

188. This ratio is the same as if we had estimated separately each of these composing ratios, and multiplied together the numbers which express these ratios. In effect, the ratio of 12 to 4 is 3, that of 25 to 5 is 5: now 3 times 5 is 15, which is the ratio of 300 to 20; and we may perceive that this is general, in observing that the ratio is measured (168) by a fraction which has the antecedent for numerator, and the consequent for denominator: therefore, the compound ratio should be a fraction having the product of the two antecedents for its numerator, and for its denominator the product of the two consequents; it is then (106) the product of the two fractions which express the two composing ratios.

189. If the ratios which we multiply are equal, the compound ratio is called *double ratio* if we have multiplied only two ratios; *triple ratio* if we have multiplied three; *quadruple* if we have multiplied four, and so on. For example, if we multiply the ratio of 2 to 3 by that of 4 to 6, which is equal to it, we shall have the compound ratio 8 : 18, which is called *double* of the ratio of 2 to 3, or of 4 to 6.

190. *If we have two proportions, and if we multiply them in order, that is to say, the first term of the one by the first term of the other, the second by the second, and so on, the four products thence resulting will be in proportion.*

For in thus multiplying two proportions, we multiply two

equal ratios by two equal ratios;(172) therefore, the two compound ratios which result should be equal, and consequently the four products should be in proportion.(172)

191. *Let us thence conclude that the squares, the cubes, and in general, similar powers of four quantities in proportion, are also in proportion*, since to form these powers we have only to multiply the proportion by itself several times successively.

192. *The square roots, cube roots, and in general, similar roots of four quantities in proportion, are also in proportion*; for the ratio of the square roots of the two first terms is nothing else than the square root of the ratio of these two terms;(142 and 167) and it is the same with regard to the ratio of the square roots of the two last terms; therefore, since the two primitive ratios are supposed equal, their square roots are equal; therefore, the ratio of the square roots of the two first terms will be equal to the ratio of the square roots of the two last. The same may be proved of the cube roots, fourth roots, etc.

USES OF THE PRECEDING PROPOSITIONS.

193. The propositions which we have demonstrated, and which we call the *Rules of proportion*, have continual application in all parts of the mathematics. We shall confine our attention here to those which belong to arithmetic, and we shall commence with those which we can deduce from what has been established,(179) which is also the basis of almost all the others.

OF THE RULE OF THREE DIRECT AND SIMPLE.

194. We distinguish several kinds of Rule of *Three*; they all have for their object to make known a term of a proportion of which three are given.

That which we call *Rule of Three Direct and Simple*, is called *simple*, because the declaration of the questions to which we apply it never contains more than four quantities, three of which are given, and the fourth is to be found.

We call it *direct*, because that of the four quantities which we there consider, there are always two which are not only relative to the two others, but which depend upon them, so

that as one of the quantities contains the other, or is contained in it, in the same manner also the quantity relative to the first contains the quantity relative to the second, or is contained in it; that is to say, in short, that a quantity and its relative may always be both antecedents or consequents, in the proportion: which is not the case in the Rule of Three Inverse, as we shall see shortly.

The method for finding the fourth term of a proportion, and, consequently, for performing the Rule of Three Direct and Simple, is sufficiently exposed;(179) but it is proper to make known, by some examples, the use which we can make of this rule.

Example I.

In a certain time, 40 workmen built 268 rods of stone wall, we demand how many rods 60 workmen would build in the same time?

It is evident that the number of rods ought to augment in proportion to the number of the workmen; so that these becoming double, triple, quadruple, etc. the first ought also to become double, triple, quadruple, etc. Thus we see that the number of rods sought ought to contain the 268 rods as often as the number 60, relative to the first, contains the number 40, relative to the second; we must then seek the fourth term of a proportion, which should commence with these three,

M. M. R.
40 : 60 :: 268 :

Or, in dividing the two first terms by 20, which may be done (170) by these other three,

R.
2 : 3 : : 268 :

Therefore, according to what has been said,(179) I multiply 268R. by 3, and I divide the product 804 by 2; which gives 402 for the quotient, and, consequently, 402R. for the number of rods that 60 workmen would build.

Example II.

A ship sailed with the same wind 275 leagues in 3 days; in what time would she sail 2000 leagues, all the other circumstances remaining the same?

It is evident that more time is required in proportion to the number of leagues, and that consequently, the number of days sought ought to contain 3 days as often as 2000 leagues contain 275 leagues: we must then seek the fourth term of a proportion which should commence with these three.

L.	L.	days
275 :	2000 : :	3 :

Multiplying 2000 by 3, and dividing the product 6000 by 275, we shall have $21\frac{5}{11}$ days for the answer.

Example III.

If a piece of land, measuring 34A. 3R. 23P. cost 168*l.* 9*s.* 4*d.* how much should we pay for 56A. 2R. 13P. at the same rate?

The price of 56A. 2R. 13P. ought to contain the price 168*l.* 9*s.* 4*d.* of the 34A. 3R. 23P. as often as 56A. 2R. 13P. contain 34A. 3R. 23P. We must then seek the fourth term of a proportion, which should commence with these three,

A.	R.	P.		A.	R.	P.			*l.*	*s.*	*d.*	
34	3	23	:	56	2	13	: :		168	9	4	:

That is to say, we must multiply 168*l.* 9*s.* 4*d.* by 56A. 2R. 13P. and divide the product by 34A. 3R. 23P. which may be done according to what has been said 122 and 128.

But it will be easier to reduce the two first terms to their least kind, that is to say, to poles; and the question will then be to seek the fourth term of a proportion, which should commence with these three others,

			l.	*s.*	*d.*	
5583	:	9053 : :	168	9	4	:

Then, multiplying 168*l.* 9*s.* 4*d.* by 9053, we shall have 1525128*l.* 14*s.* 8*d.*; and dividing by 5583, the quotient 273*l.*

3s. $5\frac{3}{5}\frac{3}{5}\frac{3}{5}\frac{3}{5}d.$ will be what we should pay for the 56A. 2R. 13P.

If there were fractions, after having reduced the two terms of the same kind to their least unit, as in this example, we should simplify the ratio of these two terms in the manner which has been taught.(171)

OF THE RULE OF THREE INVERSE AND SIMPLE.

195. The *Rule of Three Inverse and Simple* differs from the Rule of Three Direct, of which we have spoken, in this, that of the four quantities which enter into the declaration of the question for which we perform this operation, the two principals ought to contain one another, in an order entirely opposed to that of the two other quantities which are their relatives; so that, when by the examen of the question, we have given to these quantities the disposition suitable to form a proportion, one of the principal quantities and its relative form the extremes, and the other principal quantity and its relative form the means.

This does not introduce any difference in the manner of performing the operation; it is always the fourth term of a proportion which is required; or, at least, we can always bring the thing to this point.

Some arithmeticians have prescribed, for the present case, a rule subjected to the declaration of the question; we shall not follow their example; it is the nature of the question, and not its declaration, which often is faulty, that should direct in the resolution.

EXAMPLE I.

If 30 men can perform a certain work in 25 days, how many men must there be to perform the same work in 10 days?

We see, in this second case, that the number of men should be augmented in proportion to the diminution of the days; therefore, the number of men sought should contain the number 30 of the men, as many times as the number 25 of the days, relative to these, contains the number 10 of the days, relative to those. The question then is simply to find the

fourth term of a proportion which should commence with these three,

D. D. M.
10 : 25 : : 30 :

Multiplying 30 by 25, and dividing the product 750 by 10 we have 75, or 75 men, for the answer.

EXAMPLE II.

A squadron has only 15 days' provision; but the circumstances are such that it will be obliged to remain at sea 20 days; to what must they reduce the whole of the rations per day?

Let us represent by a unit the whole of the rations consumed per day; we see that the rations to which they should stint themselves should be proportionably less than this unit, as the number 20 of the days during which this economy should last, is greater than the number 15 of the days, for which they have provision; that, consequently, as often as 20 days contain 15 days, so often the whole of the provision which would have been consumed during each of these 15 days, should contain that of the provision which shall be consumed during each of the 20 days: we must then seek the fourth term of a proportion which should commence with the three following—

20 : 15 : : 1 :

This fourth term will be $\frac{15}{20}$, or $\frac{3}{4}$; they must then be reduced to $\frac{3}{4}$ of what they would have consumed per day.

OF THE COMPOUND RULE OF THREE.

196. In the two rules of Three that we have explained, the quantity sought and the quantity of the same kind which enters into the declaration of the question, have to each other a simple ratio, and this is determined by that of the two other quantities which also enter into the declaration of the question.

In the Compound Rule of Three, the ratio of the quantity sought to the quantity of the same kind, which enters into the declaration of the question, is not given by the simple ratio of two other quantities only, but by several simple

ratios, which we must compose (137) according to the examen of the question.

When once these ratios have been compounded, the rule is reduced to a rule of Three simple; the following examples will serve to elucidate what we here advance.

Example I.

Of a tract of land regularly wooded, 30 men have cleared 132 acres in 18 days, how many acres of this tract would 54 men clear in 28 days?

We see that the work depends here, not only upon the number of men, but also upon the number of days.

To have regard to both, we must consider that 30 men, who should work during 18 days, would only perform as much as 18 times 30 men, that is to say, as 540 men, who should work during one day.

In like manner, 54 men, working 28 days, would only do as much as 28 times 54 men, or 1512 men, working one day.

The question then is changed to this—540 men have cleared 132 acres of land, how many will 1512 men clear in the same time? Therefore, we must seek the fourth term of a proportion, which should commence with these three—

M. M. A.
540 : 1512 : : 132 :

or, reducing the ratio $\frac{540}{1512}$ of the two first terms to its most simple expression, (95) with these other three—

M. M. A.
5 : 14 : : 132 :

Multiplying 132A. by 14, and dividing the product by 5, we shall find 369A. 2R. 16P. for the answer to the question.

Example II.

A man, walking 7 hours per day, gives himself 30 days to travel 230 leagues; if he walked 10 hours per day, how many days would it take him to travel 600 leagues, going always at the same rate?

If he walked during the same number of hours per day, in each case, we see that he would employ a number of days

proportionably greater, as he had a greater distance to travel; but as he walks during a greater number of hours per day, in the second case, he will, for this reason, require less time; therefore, the operation relates in part to the rule of of Three direct, and to the rule of Three inverse.

We shall reduce it to a rule of Three simple, in considering that to walk during 30 days, employing 7 hours each day, is to walk during 30 times 7 hours, or 210 hours; thus, we can change the question to this other—it has required 210 hours to travel 230 leagues; how many will it require to travel 600 leagues? When we shall have found the number of hours which answers to this question, in dividing it by 10, we shall have the number of days demanded, since the man here spoken of employs 10 hours per day.

Therefore, we must seek the fourth term of a proportion, of which the three first are

L. L. H.

230 : 600 : : 210 :

We shall find that this fourth term is 547 hours and $\frac{19}{23}$, which, divided by 10, the number of the hours that this man employs each day, gives 54 days and $\frac{189}{230}$, or 54 days and $\frac{18}{23}$, for the answer.

Example III.

The foot in London being to the standard foot in France : : 15 : 16, how many feet of France will equal 720 feet of London?

It is clear, that to measure a determined length, it will require a less number of French than of English feet, in the same ratio that the first measure is, on the contrary, greater than the second; so that the question is reduced to this, viz. to calculate the fourth term of a proportion, which should commence with these three—

16 : 15 : : 720 :

multiplying 720 by 15, and dividing by 16, we shall have 675 for the number of French feet, equivalent to 720 feet in London.

Example IV.

A convoy travelling 5 hours per day, can perform a certain journey in 18 days, but we would have it to arrive in 12 days, how many hours must it travel per day ?

It is evident that this convoy should, each day, march during a number of hours as much more considerable than 5 hours, as the number 12 of the days which it must employ is, on the contrary, less than the number 18 of the days which it would have employed, if we had not forced the march. Thus the state of the question shows that we should calculate the fourth term of a proportion which should commence with these three :

12 : 18 : : 5 : or these three, 2 : 3 : : 5 :

Multiplying 5 by 3, and dividing by 2, we have $7\frac{1}{2}$ for the number of hours during which the convoy should march each day.

OF THE RULE OF FELLOWSHIP.

197. The rule of Fellowship, or Society, is thus called, because it serves to divide among several associates the benefit or the loss resulting from their society.

Its end is to divide a proposed number into parts which have to each other given ratios.

The rule which we give to effect this, is founded upon what we have established (186 :) we shall deduce it from this principle in the following example :

Example I.

Suppose, that it be required to divide 120, into three parts, which shall have to each other the same ratios as the numbers 4, 3, 2 ; the declaration of the question furnishes these two propositions :

4 : 3 : : the first part is to the second.

4 : 2 : : the first part is to the third.

Or (182) these two others :

4 is to the first part : : 3 is to the second.

4 is to the first part : : 2 is to the third.

So that we have these three equal ratios: 4 is to the first part : : 3 is to the second : : 2 is to the third.

Now, we have seen (186) that the sum of the antecedents is to the sum of the consequents, as any one antecedent is to its consequent: we can then say here, that the sum 9 of the three parts proportional to the parts that we seek, is to the sum 120 of these parts, as any one of the proportional parts is to the part of 120 which answers to it.

The rule then is, 1. to make a totality of the proportional parts given; 2. to make as many rules of Three as there are parts to find, and of which each one shall have for the first term the sum of the proportional parts given; for the second term the number proposed to divide; and for the third term one of the proportional parts given: thus, in the question that we have taken for example, we should have these three rules of Three to perform:

9 : 120 : : 4 :
9 : 120 : : 3 :
9 : 120 : : 2 :

of which we shall find (179) that the fourth terms are $55\frac{1}{3}$, 40, and $26\frac{2}{3}$, which have to each other the required ratios, and which compose in effect the number 120.

But it is easy to remark that it is not absolutely necessary to perform as many rules of Three as there are parts to find we could dispense with the last, in subtracting from the proposed number the sum of the other parts, when we have found them.

Example II.

Three persons join in trade, the first puts in stock to the amount of 20000*l.*; the second, to the amount of 60000*l.* the third, of 120000*l.*: we demand what part each should have of the gain, which is 800000*l.*, all expenses paid?

We see that it is required to divide 800000*l.* into parts which shall have to each other the same ratios as 20000, 60000, 120000; or (170) as 2, 6, 12, or 1, 3, 6, since each person should have a part in proportion to his fund; we must then add the three proportional parts, 1, 3, 6, and make the three following proportions, or only two:

10 : 800000 : : 1*l.* : the first part.
10 : 800000 : : 3*l.* : the second part.
10 : 800000 : : 6*l.* : the third part.

These three parts will be 80000*l.*, 240000*l.* and 480000*l.*

The question might be more complicated, and yet be resolved by the same principles, as in the example which follows:

Example III.

Three persons put into fellowship, the first 3000*l.*, which has been during six months in the society; the second 4000*l.* which has been in 5 months; and the third 8000*l.*, which has remained during 9 months; how much should each have of the gain, which amounts to 12050*l.*?

We shall reduce all these funds to the same time, in this manner:

The principal of 3000*l.* ought to produce during 6 months as much as 6 times 3000*l.*, or 18000*l.* during one month.

The principal of 4000*l.* should produce during 5 months as much as 5 times 4000*l.*, or 20000*l.* during one month.

Lastly, the principal of 8000*l.* should produce in 9 months as much as 9 times 8000*l.*, or 72000*l.* during one month.

Thus the question is reduced to this other: the funds of the three associates are 18000*l.*, 20000*l.*, 72000*l.*: how much should each have of the gain 12050*l.*?

In proceeding as in the above example, we shall find 1971*l.* 16*s.* $4\frac{4}{11}$*d.*, 2190*l.* 18*s.* $2\frac{2}{11}$*d.*, and 7887*l.* 5*s.* $5\frac{5}{11}$*d.*, for the answer.

Example IV.

We would distribute to Albany, to Utica, and to Rochester, a supply of utensils, namely, 4500 shovels, 4550 picks, 820 crow-bars, 800 axes, 2200 long drills, 2500 tamping irons, 2500 amalgamated priming wires, and 2500 stone hammers. This distribution should be made for each kind of utensil, proportionably and conformably to a model of a supply, by which we see that of 85000 utensils of the same kind, Albany has had 6000, Utica 1400, and Rochester 1100. We demand how many there must be of each kind for each of these places?

MODEL OF SUPPLY.

Albany - - -	6000
Utica - - -	1400
Rochester - - -	1100
	8500

Since each kind of utensil should be distributed proportionably to the numbers 6000, 1400, 1100, we shall find how many each place must have of one of the kinds; for example, of the shovels, in calculating the fourth term of each of these proportions:

8500 : 4500 or 85 : 45 or 17 : 9 : : 6000 :
17 : 9 : : 1400 :
17 : 9 : : 1100 :

We shall take the same method for calculating the number of picks, of crow-bars, etc., which ought to be distributed to each place; and we shall find that the distribution should be made as follows:

	UTENSILS.		
	For Albany.	For Utica.	For Rochester.
4500 shovels -	3177	741	582
4550 picks -	3212	749	589
820 crow-bars -	579	135	106
800 axes - -	565	132	103
2200 long drills -	1553	362	285
2500 tam. irons -	1765	412	323
2500 am. pr. wires	1765	412	323
2500 st. hammers	1765	412	323
20370	14381	3355	2634

EXAMPLE V.

Three wagoners have 1500*l.* to divide among them. The first was charged with 2 tons weight, which he conducted to 50 leagues distance; the second conducted 1½ tons 75 leagues, and the third 3 tons 60 leagues. What is the share of each?

To solve this question by the preceding rule, we must re-

duce these different transports to the same distance, in this manner:

2 tons carried 50 leagues ought to be paid as much as 50 times 2 tons, or 100 tons carried one league. In like manner, $1\frac{1}{2}$ tons carried 75 leagues should be paid as much as 75 times $1\frac{1}{2}$ tons, or $112\frac{1}{2}$ tons carried one league. Lastly, 3 tons carried 60 leagues should be paid as much as 60 times 3 tons, or 180 tons carried one league.

Thus the question is the same as if the three wagoners had carried to the same distance, the first 100 tons, the second $112\frac{1}{2}$ tons, and the third 180 tons. It is required then to divide 1500*l.* into three parts, which shall have to each other the same ratio as the numbers 100, $112\frac{1}{2}$, and 180, in calculating the fourth term of each of the following proportions:

$392\frac{1}{2}$: 1509 : : 100	: 382*l.*	3*s.*	4*d.*
$392\frac{1}{2}$: 1500 : : $112\frac{1}{2}$	: 429	18	9
$392\frac{1}{2}$: 1500 : : 180	: 687	17	11

The share of the first is 382*l.* 3*s.* 4*d.*; of the second 429*l.* 18*s.* 9*d.*; and of the third 678*l.* 17*s.* 11*d.*

REMARK UPON THE SUBJECT OF THE PRECEDING RULE.

198. It will be useful to examine a case which might embarrass beginners. If we proposed this question—to divide 650 into three parts, of which the first shall be to the second : : 5 : 4, and the first to the third : : 7 : 3.

We cannot here apply the preceding rule without a preparation, which consists in rendering the same, in each given ratio, the part proportional to one of the three parts sought; for example, that of the first; this is easily done in multiplying the two terms of each ratio by the first term of the other ratio: thus the two ratios 5 : 4 and 7 : 3 will be brought to have the same first term in multiplying the two terms of the first by 7, and the two terms of the second by 5, which does not alter their value,(170) and gives the ratios 35 : 28 and 35 : 15, so that the question becomes, to divide 650, into three parts, which shall be to each other as the numbers 35, 28, and 15, which will easily be done by the preceding rule.

If we were required to divide a number into four parts, of

which the first should be to the second : : 5 : 4, the first to the third : : 9 : 5, and the first to the fourth : : 7 : 3, we should reduce these ratios to have the same first term, in multiplying the two terms of each by the product of the first terms of the two others; thus in this example, we should change these three ratios to these other three, 315 : 252, 315 : 175, 315 : 135; so that the question is changed to this, viz. to divide the proposed number into four parts, which shall be to each other as the numbers 315, 252, 175, and 135.

OF SOME OTHER RULES DEPENDING UPON PROPORTION.

199. Although the following rules be of less frequent use than the preceding, we cannot however entirely omit them; for besides that they are not without utility by themselves, they are otherwise useful in making known the extent of the uses of proportions.

200. The first of which we shall speak is the rule of *a false position*. We often apply it to resolve questions which belong to the Rule of Fellowship, from which it differs in this, viz. that instead of taking the proportional parts such as they are given by the declaration of the question, it takes one arbitrarily, and subordinates the others to it conformably to the question, which renders the calculation somewhat more easy.

EXAMPLE I.

Divide $640 between three persons, so that the second may have four times as much as the first, and the third twice and $\frac{1}{3}$ as much as the two others together.

I take arbitrarily, to represent the first part, the number 3, of which I can conveniently take the $\frac{1}{3}$.

The first part being 3, the second will be 12, and the third will be 35.

The question then is to divide 640 into three parts, which may be to each other as the three numbers 3, 12, 35, which will be done as has been said.(197)

The rule of a false position serves also to resolve questions which are in some sort the reverse of the rule of Fellowship, since it requires from the sum of certain parts of a number to find the number itself, as in the example which follows:

Example II.

We would find a number of which the $\frac{1}{3}$, the $\frac{1}{5}$, and the $\frac{3}{7}$, make 808.

I take a number of which I can conveniently have the $\frac{1}{3}$, the $\frac{1}{5}$, and the $\frac{3}{7}$, which is easy in multiplying the three denominators. This number will be 105; I take the $\frac{1}{3}$ of it which is 35, the $\frac{1}{5}$ which is 21, and the $\frac{3}{7}$ which is 45. I add these three numbers, and I have 101, which is composed of the parts of 105 in the same manner that 808 is of those of the number in question: therefore, the number in question ought to have the same relation to 808 that 105 has to 101; it ought then to be the fourth term of a proportion which commences with these three:

101 : 105 : : 808 : or these 1 : 105 : : 8 :

This fourth term is 840, of which 808 contains, in effect, the $\frac{1}{3}$, the $\frac{1}{5}$, and the $\frac{3}{7}$.

201. The second rule of which we shall speak is that of two false positions.

It serves in questions where it is required to divide, not the proposed number, but only a part of this number, into parts proportional to given numbers; the following example will make known the rule and its use.

Example.

It is required to divide $6954 between three persons, so that the second may have as much as the first and $54 more, and the third as much as the two others together and $78 more.

Without the $54 and the $78, it is clear that we should only have to divide the proposed number into parts proportional to the number 1, 1, and 2; but since we must first take from the sum $54 for the second person, and $54 plus $78 for the third, it is evident that there is but a part of the proposed number which we must divide into parts proportional to 1, 1 and 2: as this part, which is easy to find in the present example, may be difficult to perceive in other circumstances, we pursue the following method:

Suppose, for the first part, any number at pleasure, for example $1, the second part will be $1 plus $54, that is to say, $55, and the third will be $1 plus $55 plus $78, that is to say, $134: the sum of these parts is $190.

If it had only been required to divide into parts proportional to 1, 1 and 2, the first part being always supposed $1, the second would be $1, the third would be $2, and the sum would be $4, the difference of which from $190, that is to say, $186, is that which must first be taken from the proposed sum $6954, which reduces it to $6768: it remains then to divide $6768 into parts proportional to 1, 1 and 2, according to the above rules: and having found that the first part is $1692, we shall thence conclude that the other parts are $1746 and $3316; in effect the whole of these three parts is $6954.

202. We shall also find, among arithmeticians, many other rules which are nothing else than the application of the Rules of Three to different questions, such as the questions of *interest*, of *change*, of *discount*, etc.

We will not enter into these details, which cannot be difficult to those who, having well understood the principles established above, will have at the same time the state of the question present to the mind. We will limit ourselves to a single example.

A person gave a merchant a note for £2854, payable in a year; he comes to discharge his note at the end of 7 months, and the merchant consents to deduct for the 5 remaining months the interest which was comprised in the note, at the rate of 6 per cent. for 12 months; for what sum should the merchant give up the note?

Since 12 months produce £6 for the interest of 100, 7 months should produce an interest which we shall find in seeking the fourth term of a proportion, of which the three first are

$$12 : 7 :: 6 :$$

This fourth term is $3\frac{1}{2}$. Now when the interest was taken at 6 per 100, we counted for £106 what was only worth 100; therefore, when the interest is at $3\frac{1}{2}$, we count for $103\frac{1}{2}$ that which is only worth 100; it follows then that the sum which should have been paid 106 should now be paid only $103\frac{1}{2}$.

Therefore, the sum sought should be the fourth term of a proportion, of which the three first are

$$106 : 103\tfrac{1}{2} :: 2854 :$$

This fourth term, which is 2786*l.* 13*s.* $9\frac{15}{53}$*d.* is the sum that the debtor should give to withdraw his note.

OF THE RULE OF ALLIGATION OR MIXTURE.

203. The questions which belong to this rule are of two kinds. In the one, it is required to find the mean value of several kinds of things, the number and particular value of each of which are known.

In the second, it is required to find the quantities of each kind of thing which enters into one or more mixtures, when we know the price or value of each kind, and the price or the total value of each mixture.

We will reserve the questions of the second kind for algebra.

As to the questions of the first, the rule for resolving them is as follows:

Multiply the value of each kind of thing, by the number of the things of that kind; add all the products, and divide the sum by the total number of the things of all the kinds.

EXAMPLE.

We employ 200 workmen, of whom 50 are paid at the rate of 40 shillings per day, 70 at the rate of 30 shillings, 50 at the rate of 25 shillings, and 30 at the rate of 20 shillings: what should each workman have per day, the one bearing the other?

Fifty workmen at	40*s.* per day, make an expense of	2000*s.*
Seventy, at	30*s.* - - -	2100
Fifty, at -	25*s.* - - - -	1250
Thirty, at -	20*s.* - - -	600
		5950*s.*

The expense of 200 workmen is then 5950 shillings per day, and consequently, in dividing by 200, each workman, one bearing the other, should have 29*s.* 9*d.* per day. The

other questions of this kind are so easy to resolve after this example, that we think proper not to insist upon the subject.

OF ARITHMETICAL PROGRESSIONS.

204. The arithmetical progression is a series of terms, each of which surpasses the preceding term, or is surpassed by it by the same quantity.

For example, this series:

÷ 1 : 4 . 7 . 10 . 13 . 16 . 19 . 22 . 25, etc.

is an arithmetical progression, because each of its terms surpasses that which precedes it, by the same quantity, which is in this case 3.

The two points separated by a bar, which we see here at the head of the progression, are destined to show that in reading this progression, we should repeat each term, except the first and the last, in this manner:

1 *is to* 4 *as* 4 *is to* 7 *as* 7 *is to* 10, etc.

The progression is called *increasing*, or *decreasing*, according as the terms go on increasing or decreasing; but as the properties of both are the same, in simply changing the words *plus* to *minus*, *add* to *subtract*, and *multiply* to *divide*, we shall here consider it only as increasing.

205. We see then, according to the definition of the arithmetical progression, that with the first term and the common difference, or the ratio of the progression, we can form all the other terms, in successively adding this ratio; and that consequently:

The second term is composed of the first, plus the ratio.

The third is composed of the second, plus the ratio, and consequently of the first, plus twice the ratio.

The fourth is composed of the third, plus the ratio, and consequently of the first, plus three times the ratio, and so on for any number of terms.

206. So that we can say, universally, that *any term of an arithmetical progression, is composed of the first, plus as many times the ratio as there are terms before it.*

207. Therefore, if the first term be a cipher, every

other term of the progression will be equal to as many times the ratio as there are terms before it.

* If we call ω the greatest term of an arithmetical progression, a the least term, d the difference, and n the number of the terms, we shall have $\omega = a + d \times (n - 1.)$

208. This principle has the two following applications:

1. It serves to find any term of a progression, without our being obliged to calculate those which precede it. Suppose we demand, for example, what would be the 100th term of the following progression:

$$+4 . 9 . 14 . 19 . 24, \text{etc.}$$

Since the term sought is the hundredth, it has then 99 terms before it; it is then composed of the first term 4, and of 99 times the ratio 5; it is then 4 plus 495, that is to say, 499.

209. 2. This same principle serves to connect any two numbers by a succession of as many other numbers as we would, so that the whole may form an arithmetical progression; this we call *inserting* between two given numbers several *arithmetical mean proportionals*, or simply, several *arithmetical means*.

For example, we can connect 1 and 7, by 5 numbers which make an arithmetical progression with 1 and 7; these numbers are 2, 3, 4, 5, 6; but as it is not always easy to determine, at first sight, what these numbers should be, we can find them by the aid of the principle which we have established, as follows:

All that is required is to find the ratio which should reign in this progression.

Now as the greatest of the two proposed numbers is to be the last term of the progression, it should be composed of the first term, that is to say, of the least of the two given numbers, plus as many times the ratio as there are terms before it; therefore, if from the greatest of these two numbers we subtract the least, the remainder will be composed of as many times the ratio as there should be terms before the greatest; therefore (74) if we divide this remainder by the number of the terms which should precede the greatest, we shall have this ratio.

Now the number of the terms which should precede the greatest, is greater by a unit than the number of the means which we would insert between the two; therefore *to insert between two given numbers as many arithmetical means as we would, we must subtract the least of these two numbers from the greatest, and divide the remainder by the number of the means augmented by a unit.* The quotient will be the difference or the ratio which should reign in the progression.

Having the greatest term of an arithmetical progression, the least term, and the number of the terms, to find the difference.

Since the greatest term is equal to the least term, plus the difference multiplied by the number of the terms diminished by a unit, it is evident that in subtracting the least term from each of these equalities, we shall have:

The greatest term minus the least, equal to the difference multiplied by the number of the terms diminished by a unit, that is to say, that the difference multiplied by the number of the terms diminished by a unit, is equal to the greatest term minus the least, that is to say, the difference of the extremes. Dividing the two equalities by the number of the terms, diminished by a unit, we shall have the difference equal to the difference of the extremes divided by the number of the terms diminished by a unit; that is to say:

$$d=\frac{a-a}{n-1}.$$

For example, if between 4 and 11 we require the insertion of 8 arithmetical means, I subtract 4 from 11, there remains 7, which I divide by 9, the number of means augmented by a unit; the quotient $\frac{7}{9}$ is the difference which should reign in the progression, which will, consequently, be

$$\div 4 \,.\, 4\tfrac{7}{9} \,.\, 5\tfrac{5}{9} \,.\, 6\tfrac{3}{9} \,.\, 7\tfrac{1}{9} \,.\, 7\tfrac{8}{9} \,.\, 8\tfrac{6}{9} \,.\, 9\tfrac{4}{9} \,.\, 10\tfrac{2}{9} \,.\, 11.$$

In like manner, if we require nine arithmetical means between 0 and 1, subtracting 0 from 1, there remains 1, which we must divide by 10, the number of the means augmented by a unit, which gives $\frac{1}{10}$, or 0,1, for the ratio. And, consequently, the progression will be

$$\div 0 \,.\, 0{,}1 \,.\, 0{,}2 \,.\, 0{,}3 \,.\, 0{,}4 \,.\, 0{,}5 \,.\, 0{,}6 \,.\, 0{,}7 \,.\, 0{,}8 \,.\, 0{,}9 \,.\, 1.$$

210. We see from this, that between any two numbers,

however near they may be to each other, we can always insert as many arithmetical means as we would.

We will say no more on the arithmetical progressions, which we only speak of here with regard to logarithms; we shall have occasion to return to them elsewhere.

* The sum of the terms of an arithmetical progression is equal to half the sum of the extremes multiplied by the number of terms.

Let us take, for example, the arithmetical progression

$$\div 3 . 5 . 7 . 9 . 11 . 13.$$

Let us write underneath this progression, this same progression inverted, in the following manner—

$$\div 3 . 5 . 7 . 9 . 11 . 13,$$
$$\div 13 . 11 . 9 . 7 . 5 . 3.$$

It is evident that the sums of the corresponding terms, 3 plus 13, 5 plus 11, 7 plus 9, etc. are equal to one another; and that the sum of the two progressions is equal to the sum of two corresponding terms, multiplied by the number of the terms of one of these two progressions. But the sum of two corresponding terms is equal to the sum of the extremes; therefore, the sum of the two progressions is equal to the sum of the extremes of one of the two progressions multiplied by the number of the terms of one of them. Therefore, the sum of the terms of one of these progressions is equal to half the sum of the extremes, multiplied by the number of its terms.

The reasoning would be the same for any other arithmetical progression; therefore, the sum of the terms of an arithmetical progression is equal to half the sum of the extremes multiplied by the number of the terms.

If we call s the sum of the terms, we shall then have

$$s=\frac{\omega+a}{2}\times n.$$

It is by the aid of these two equalities

$$\omega=a+d(n-1),$$
$$s=\frac{\omega+a}{2}\times n,$$

that the following table is constructed, by means of which, knowing three of the five things a, ω, d, n, s, of an arithmetical progression, we can always find the other two.

Being given.	To find,	FORMULÆ.
ω, d, n	a	$a=\omega-dn+d.$
ω, n, s		$a=\frac{2s}{n}-\omega.$
ω, d, s		$a=\frac{1}{2}d+\sqrt{(-2ds+\frac{1}{4}d^2+\omega d+\omega^2)}.$
d, n, s		$a=\frac{s}{n}-\frac{dn+d}{2}.$
a, d, n	ω	$\omega=a+dn-d.$
a, n, s		$\omega=\frac{2s}{n}-a.$
a, d, s		$\omega=\frac{1}{2}d+\sqrt{(2ds+\frac{1}{4}d^2-ad+a^2)}.$
d, n, s		$\omega=\frac{s}{n}+\frac{dn-d}{2}.$
a, ω, n	d	$d=\frac{\omega-a}{n-1}.$
a, n, s		$d=\frac{2s-2an}{nn-n}.$
a, ω, s		$d=\frac{\omega^2-a^2}{2s-a-\omega}.$
ω, n, s		$d=\frac{2\omega n-2s}{nn-n}.$
a, ω, d	n	$n=1+\frac{\omega-a}{d}.$
a, ω, s		$n=\frac{2s}{a+\omega}.$
a, d, s		$n=\frac{1}{2}-\frac{a}{d}+\sqrt{\left(\frac{2s}{d}+\frac{1}{4}-\frac{a}{d}+\frac{a^2}{d^2}\right)}.$
ω, d, s		$n=-\frac{1}{2}+\frac{\omega}{d}+\sqrt{\left(-\frac{2s}{d}+\frac{1}{4}+\frac{\omega}{d}+\frac{\omega^2}{d^2}\right)}.$
a, ω, n	s	$s=\frac{an+\omega n}{2}.$
a, d, n		$s=an+\frac{dnn-dn}{2}.$
a, d, ω		$s=\frac{a+\omega}{2}+\frac{\omega^2-a^2}{2d}.$
ω, d, n		$s=\omega n-\frac{dnn-dn}{2}.$

OF GEOMETRICAL PROGRESSION.

211. The geometrical progression is a series of terms of which each contains that which precedes it, or is contained in it, the same number of times; for example, this series—

$$\div\!\div\ 3 : 6 : 12 : 24 : 48 : 96 : 192,$$

is a geometrical progression, because each term contains that which precedes it the same number of times, which is here 2.

This number of times is what we call *the ratio* of the progression.

The four points which precede the progression have the same signification as the two points which precede the arithmetical progression.(204) But we put four of them to signify that the progression is geometrical.

The progression is called increasing or decreasing, according as its terms go on augmenting or diminishing.

We shall always consider the geometrical progression as increasing, because the properties are the same in both, in changing the words *to multiply* to those of *to divide*, and the words *to contain* to those of *to be contained*.

Since the second term contains the first as many times as there are units in the ratio, it is composed of the first multiplied by the ratio.

Since the third term contains the second as many times as there are units in the ratio, it is composed of the second multiplied by the ratio, and consequently of the first multiplied by the ratio, and again multiplied by the ratio; that is to say, of the first multiplied by the square, or second power of the ratio.

Since the fourth term contains the third as many times as there are units in the ratio, it is composed of the third term multiplied by the ratio, and consequently of the first multiplied by the square of the ratio, and again multiplied by the ratio; that is to say, multiplied by the cube or third power of the ratio.

For example, *in the above progression*, 6 is composed of the first term 3 multiplied by the ratio 2; 12 is composed of the

first term 3, multiplied by the square 4 of the ratio 2; 24 is composed of the first term 3 multiplied by the cube 8 of the ratio 2.

212. In continuing the same reasoning, we see that any *term of a geometrical progression is composed of the first multiplied by the ratio raised to a power expressed by the number of the terms which precede that term.*

Therefore, if the first term of the progression be a unit, each of the other terms will be formed of the ratio itself raised to a power signified by the number of the terms which precede it; for, the multiplication by the first term which is a unit, does not augment the product.

* If we call u the greatest term, a the least, and q the quotient or ratio, we shall have:

$$u = a \times q^{n-1}.$$

To raise a number to a proposed power, to the seventh, for example, we must, according to the idea that we have given of powers, multiply this number by itself six successive times. Thus, to raise 2 to the seventh power, I should say, twice 2 are 4, twice 4 are 8, twice 8 are 16, twice 16 are 32, twice 32 are 64, twice 64 are 128, which is the seventh power of 2; but we can abridge the operation in divers ways: for example, I can first square 2, which gives 4; cube this 4, which gives 64, and multiply 64 by 2, which gives 128; or I can cube 2, which gives 8; square 8, which gives 64, and multiply 64 by 2, which makes 128; in a word, it is of little importance what method we take, provided that 2 be found 7 times factor in the product.

213. The principle that we have established (212) upon the formation of any term in a geometrical progression, and the remark that we have made, may serve to calculate any term whatever of the progression, without being obliged to calculate those which precede that term. If we demand, for example, what would be the twelfth term of the progression

$$\div\!\!\div\ 3 : 6 : 12 : 24 : \text{etc.}$$

As I know (212) that this twelfth term should be composed of the first, multiplied by the ratio raised to a power ex-

pressed by the number of the terms which precede this twelfth, I see that, to form it, I must multiply 3 by the eleventh power of the ratio 2. To form this eleventh power, I cube 2, which gives me 8; I cube 8, which gives me 512 for the ninth power, and lastly, I multiply 512, ninth power of the ratio, by 4, the second power, and I have 2048 for the eleventh power of 2; I then multiply 2048 by 3, and I have 6144 for the twelfth term of the progression.

When we multiply any power of a number by any other power of that number, the sum of the indices of these two powers, is the index of the power that the number is raised to by this multiplication. Thus, $3^2 \times 3^3 = 3^5$, that is to say, $9 \times 27 = 243$, which is the fifth power of 3.

214. Another application which we can make of this same principle, is to find as many geometrical means as we wish between two given numbers. If we required three geometrical means between 4 and 64, with a little attention, we should see that these three geometrical means are 8, 16, 32. In effect, ∺ 4 : 8 : 16 : 32 : 64, forms a geometrical progression; but if we proposed other numbers than 4 and 64, or if we required any other number of geometrical means, we should not find them so easily.

We find them by virtue of the principle in question, as follows:

The question requires to find the ratio which should reign in the progression, because when it shall be found, we shall easily form the terms in successively multiplying by this ratio.

Let it be required, for example, to find nine geometrical means between 2 and 2048: 2048 will then be the last term of a geometrical progression which commences with 2, and which ought to have nine terms between the first and the last: 2048 is then composed of the first term 2, multiplied by the ratio raised to a power expressed by the number of the terms which should precede 2048: hence (69) if we divide 2048 by the first term, the quotient will be the ratio raised to a power signified by the number of terms which ought to precede 2048; therefore, in seeking the root of this power, we shall have the ratio: now this power should be the tenth, for, as there must be nine terms between 2 and 2048, there

are consequently ten before 2048: therefore we must extract the tenth root of the quotient found in dividing the greatest number 2048, by the least number 2.

215. As we can apply the same reasoning in every case, let us conclude in general that, *to insert between two given numbers, as many geometrical means as we wish, we must divide the greatest of these two numbers by the least, which will give a quotient; we shall extract of this quotient a root signified by the number of the means augmented by a unit.*

Therefore, to return to our example, I divide 2048 by 2; this gives me 1024, of which I seek the tenth root,(†) which is 2. Thus, to form the means in question, I multiply the first term 2 continually by the ratio 2; and after having formed nine means, I again fall upon the number 2048, as we see here:

÷ 2 : 4 : 8 : 16 : 32 : 64 : 128 : 256 : 512 : 1024 : 2048.

In like manner, if it were required to find four geometrical means between 6 and 48, I should divide 48 by 6, and should extract the fifth root of the quotient 8; as 8 has not an exact fifth root, we can never assign exactly, in numbers, four geometrical means between 6 and 48; but we can approach to this root as near as we wish by a method analogous to that of the square and cube roots, which we shall make known in Algebra. In the mean time, it is sufficient to conceive that it is possible to find a number which, multiplied four times successsively by itself, approaches nearer and nearer to reproduce 8; and that it is the same for any other number and any other root; and hence we shall conclude that between any two numbers we can always find as many geometrical means as we would, either exactly or by an approximation carried to any degree of exactness, and this is all that is necessary to enable us to proceed to Logarithms.

(†) We have not given a method for extracting the tenth root of a number, but it is the same with this as with the square and cube roots; the square root cannot have more than one figure when the proposed number has not more than two: the cube root ought to have but one figure when the proposed number has not more than 3; in like manner, the tenth root will never have more than one figure when the proposed number has not more than ten: it is the same with the other roots; the thirtieth, for example, will only have one figure, if the proposed number has not more than thirty figures; this is demon-
* has been done for the square and cube roots.

* The sum of the terms of a geometrical progression is equal to the product of the greatest term, multiplied by the quotient, minus the least term, and the whole divided by the quotient minus 1; calling s the sum of the terms, we shall have:

$$s=\frac{\omega\times q-a}{q-1}.$$

Let us take, for example, the following geometrical progression:

$$\div\ 2:6:18:54:162\text{(A.)}$$

which we may write thus:

$$2:6::6:18::18:54::54:162\text{(B.)}$$

But in B. the sum of the antecedents is to the sum of the consequents as one antecedent is to its consequent. (Bezout, No. 185.) Therefore, $2+6+18+54:6+18+54+162::2:6$.

Now it is evident that $2+6+18+54$ is equal to the sum of the terms of the progression A, minus the last term 162, and that $6+18+54+162$ is equal to the sum of all the terms of the first progression, minus the first; it is then evident, that if we call s the sum of all the terms of the progression A, a its first term, its last, q its quotient, we shall have:

$$s-\omega:s-a::a:aq$$

Therefore, $(s-\omega)\times aq=(s-a)\times a$.

Performing the multiplications indicated, and taking the value of s, we shall find:

$$s=\frac{\omega\times q-a}{q-1}.$$

It is by the aid of these two equations

$$\omega=a\times q^{n-1}$$

$$s=\frac{\omega\times q-a}{q-1}$$

that the following table is calculated, by means of which, knowing three of the five things ω, a, q, n, s, of a geometrical progression, we can always find the two others.

It is by the aid of this table that we execute all the most difficult calculations of the bank and the finances, such as compound interest, annuities, etc.

Being given,	To find,	FORMULÆ.
a, q, n	a	$a=\frac{\omega}{q^{n-1}}.$
ω, s, n		$(s-a)\frac{1}{a^{n-1}}=(s-\omega)\frac{1}{\omega^{n-1}}.$
ω, q, s		$a=\omega q-sq+s.$
q, n, s		$a=s\left(\frac{q-1}{q^n-1}\right).$
a, q, n	ω	$\omega=aq^{n-1}.$
a, s, n		$(s-\omega)\frac{1}{\omega^{n-1}}=(s-a)\frac{1}{a^{n-1}}.$
a, q, s		$\omega=s-\frac{s+a}{q}.$
q, n, s		$\omega=sq^{n-1}\left(\frac{q-1}{q^n-1}\right).$
a, ω, n	q	$q=\pm\sqrt[n-1]{\frac{\omega}{a}}.$
a, n, s		$q^n-\frac{s}{a}q+\frac{s}{a}-1=0.$
a, ω, s		$q=\frac{s-a}{s-\omega}.$
n, ω, s		$q^n-\frac{s}{s-\omega}q^{n-1}+\frac{\omega}{s-\omega}=0.$
a, ω, q	n	$n=1+\frac{L\omega-La}{Lq}.$
a, ω, s		$n=1+\frac{L\omega-La}{L(s-a)-L(s-\omega)}.$
a, q, s		$n=\frac{L(sq-s+a)-La}{Lq}.$
ω, q, s		$n=1+\frac{L\omega-L(aq-sq+s)}{Lq}.$
a, ω, n	s	$s=\frac{\frac{n}{\omega^{n-1}}-\frac{n}{a^{n-1}}}{\frac{1}{\omega^{n-1}}-\frac{1}{a^{n-1}}}.$
a, q, n		$s=a\left(\frac{q^n-1}{q-1}\right).$
a, ω, q		$s=\frac{\omega q-a}{q-1}.$
ω, n, q		$s=\frac{\omega}{q^{n-1}}\left(\frac{q^n-1}{q-1}\right).$

OF LOGARITHMS.

216. *Logarithms* are numbers in arithmetical progression, which answer, term for term, to a like series of numbers in geometrical progression. If we take, for example, the following geometrical and arithmetical progressions—

∺ 2 : 4 : 8 : 16 : 32 : 64 : 128 : 256, etc.
÷ 3 . 5 . 7 . 9 . 11 . 13 . 15 . 17, etc.

Each term of the lower series is called the logarithm of the term which is in the same place in the upper series.

217. The same number may then have an infinity of different logarithms, since to the same geometrical progression we can make correspond an infinity of different arithmetical progressions. As we shall here consider logarithms only with regard to the use which we can make of them in numerical calculations, we will not stay to consider the different geometrical and arithmetical progressions, which we might compare to each other; we shall directly proceed to those which have been considered in the formation of the tables of logarithms.

218. The decuple progression is chosen for the geometrical progression, and the natural series of numbers for the arithmetical progression; that is to say, the two following progressions—

∺ 1 : 10 : 100 : 1000 : 10000 : 100000 : 1000000
÷ 0 . 1 . 2 . 3 . 4 . 5 . 6.

219. Thus, it is easy to perceive that the logarithm of a unit followed by any number of ciphers has always as many units as there are ciphers on the right of the unit.

We shall not here teach the method which was followed to find the intermediate logarithms of the decuple progression; it depends upon principles that we cannot here explain; but we shall show their formation in a way, which, truly, would not be the most expeditious for calculating these logarithms, but which suffices as well for the understanding of this formation as to give a reason for the uses to which we apply these artificial numbers.

220. According to the definition that we have given of logarithms, we see that to have the logarithm of any number,

of 3, for example, this number must make a part of the fundamental geometrical progression. Now, although we do not see that 3 can make a part of the geometrical progression ÷ 1 : 10 : 100, etc.; we see, however, that if, between 1 and 10, we should insert an exceeding great number of geometrical means,(214) as we should then ascend from 1 to 10 by degrees, so much the more compact as the number of these means should be greater, one of two things would happen; either that some one of these means would be found precisely the number 3, or that there should be found two successive numbers, between which the number 3 would be comprised, and the difference of each of which from 3 would be less in proportion, as the number of the means inserted should be greater.

This established, if we should insert, in like manner, between 0 and 1 as many arithmetical means as we have inserted geometrical means between 1 and 10, each term of the geometrical progression having for logarithm the corresponding term of the arithmetical progression, we should take in this, for the logarithm of 3, the number which should here be found in the same place as 3 is found in the geometrical progression; or, if 3 were not exactly one of the terms of this, we should take, in the arithmetical progression, the term answering to that in the geometrical progression, which approaches the nearest to the number 3.

It is thus that we should proceed, in effect, if we had not more expeditious means. At any rate, this is the amount of the calculation of logarithms.

221. We must then represent that having inserted 1000000 geometrical means between 1 and 10, a like number between 10 and 100, a like number between 100 and 1000, etc. we have also inserted the same number of arithmetical means between 0 and 1, the same number between 1 and 2, the same number between 2 and 3 etc.; that having ranged all the first in the same line, and all the second underneath, we have sought in the first the number nearest to 2, and have taken in the lower series the corresponding number; that we have likewise sought in the first the number nearest to 3, and that

we have taken in the lower series the corresponding number; that we have done the same successively, for the numbers 4, 5, 6, etc.; that, finally, having transported into the same column, as we see in the table, the numbers 1, 2, 3, 4, 5, etc. we have written in the adjacent column the terms of the arithmetical progression corresponding to these, or at least, those which approach the nearest to them; then we shall have the idea of the formation of logarithms, and of their disposition in the ordinary tables.

The table at the end of the book contains only the decimal part of the logarithm of each number as far as 10000, except the part from 1 to 100.

222. Let us remark, with regard to this Table, that the first figure which should stand on the left of each logarithm, that is to say, the figure on the left of the comma, is called the *index*, because it indicates in what decade the number is comprised to which this logarithm belongs; for example, if the number be between 1000 and 10000, because the logarithm of 1000 is 3, and that of 10000 being 4, any number between 1000 and 10000 can only have for its logarithm 3 and a fraction; the logarithm has then 3 for its index, and the other figures express this fraction reduced to decimals.

From what we have here said, it is plain that *the index of a logarithm is always one less than the number of figures contained in the whole number.*

Hence, the index of 116 is 2, being one less than the number of figures contained in 116; also, the index of 126,3 is 2, for although this number contains four figures, yet there are only three belonging to the whole number, the figure 3 being a decimal.

To find the logarithm of a number consisting of three figures, I seek this number in the left hand column of the table, and opposite to it, in the column marked 0 at the top or bottom, I find the decimal part of its logarithm, to which I prefix the index as signified above: thus the logarithm of 276 is 2,440909.

To find the logarithm of a number consisting of four figures, seek the first three figures in the left hand column; opposite to these, in the column marked with the fourth figure

at the top or bottom, is the decimal part of the logarithm required, to which prefix the index as before: thus the log. of 1263 is 3,101403; that of 126,3 is 2,101403; and that of 12,63 is 1,101403.

PROPERTIES OF LOGARITHMS.

223. As we here only treat of logarithms such as they are in the ordinary tables, the properties we are about to explain relate only to the geometrical progressions which have unity for the first term, and the arithmetical progressions which have a cipher for the first term.

Let us then compare, term for term, any geometrical progression, having a unit for the first term, with any arithmetical progression having a cipher for the first term; for example, the two following progressions:

÷÷ 1 : 3 : 9 : 27 : 81 : 243 : 729 : 2187 : 6561, etc.

÷ 0 . 4 . 8 : 12 . 16 . 20 . 24 . 28 . 32, etc.

It follows from the nature and perfect correspondence of these two progressions, that as many times as the ratio of the first is factor in any one of the terms of that progression, so many times the ratio of the second is contained in the corresponding term of the second; for example, in the term 2187, the ratio 3 is 7 times factor, and in the term 28, the ratio 4 is contained seven times.

In effect, according to what has been said (206 and 212) the ratio is factor in any term whatever of the first as many times as there are terms preceding it; and in the second, any term whatever is composed of as many times the ratio as there are terms preceding it. Now there is the same number of terms on the one hand and on the other.

Let us thence conclude, that any term of the geometrical progression will always have for its correspondent in the arithmetical progression, a term that will contain the ratio of this as many times as the ratio of the first is factor in the term first mentioned.

224. Therefore, *if we multiply the one by the other two terms of the geometrical progression, and if we add at the same time the two corresponding terms of the arithmetical progression,*

the product and the sum will be two terms that will correspond with each other in these progressions.

For it is evident that the ratio will be factor in the product as many times as it is in both the terms multiplied; and that the ratio in the arithmetical progression will be contained in the sum as many times as it is in both the terms added.

225. Therefore, we can, by the addition alone of two terms of the arithmetical progression, find the product of the two corresponding terms of the geometrical progression, supposing these two progressions to be sufficiently extended.

For example, in adding the two terms 8 and 24, which correspond with 9 and 729, I have 32, which answers to 6561; whence I conclude that the product of 729 by 9 is 6561, which it is in effect.

226. Therefore, since the natural numbers which compose the first column of the table of logarithms, have been taken from a geometrical progression which commences with a unit; and since their logarithms are the corresponding terms of an arithmetical progression which commences with a cipher, we shall conclude that *in adding the logarithms of two numbers, we have the logarithm of their product.*

USE OF LOGARITHMS.

227. To perform a multiplication by logarithms, we must add the logarithm of the multiplicand to the logarithm of the multiplier; the sum will be the logarithm of the product; wherefore, seeking this sum among the logarithms of the table, we shall find the product opposite; for example, if we propose to multiply 14 by 13, I find in the table that the

logarithm of 14 is	- -	1,146128
and that the logarithm of 13 is	-	1,113943
The sum total	-	2,260071

answers in the same table to the number 182, which is in effect the product.

228. To square a number, it is sufficient then to double its logarithm, since we should add this logarithm to itself in order to multiply the number by itself.

229. For the same reason, to cube a number, we must

sign which shows that the subtraction has not been entirely performed. This is the sign minus —. Thus the logarithm of the fraction $\frac{11}{81}$ is —0,917648.†

236. This sign is intended to show in the calculation, that the logarithms of fractions should be employed according to a rule entirely opposed to that which we have prescribed for the logarithms of whole numbers, or of whole numbers joined to fractions; that is to say, that if we have to multiply by a fraction, we must subtract the logarithm of this fraction; if, on the contrary, we have to divide by a fraction, we must add its logarithm.

The reason of this is, for the multiplication, that to multiply by a fraction is to multiply by the numerator, and to divide afterward by the denominator; therefore, when we operate by logarithms, we should add the logarithm of the numerator, and afterward subtract that of the denominator, or, which amounts to the same thing, we should only subtract the excess of the logarithm of the denominator above that of the numerator: now, this excess is precisely the logarithm of the fraction. With regard to the division, the reason is also easy to comprehend. In effect, to divide by $\frac{3}{4}$ is (109) to multiply by $\frac{4}{3}$; therefore, in operating by logarithms, we must add the logarithm of $\frac{4}{3}$, that is to say,(234) the difference between the logarithm of 4 and the logarithm of 3, or between the logarithm of the denominator of the proposed fraction and the logarithm of its numerator.

237. It often happens, that in converting to a single fraction the whole number and the fraction, the logarithm of which we seek, the numerator is a number which passes the limits of the table. For example, if we would have the logarithm of $53\frac{381}{3704}$, this number reduced to a fraction becomes $\frac{196381}{3704}$, the numerator of which passes the limits of the most extensive tables.

It is then proper to know how we can find the logarithm of a number which passes these limits.

† To consider negative numbers as numbers less than nothing, is to have a very false idea, as there can be no number less than nothing.

The method we are about to give is not rigorously exact, but it is more than sufficiently so for ordinary purposes. Before we expose it, let us observe:

238. 1. That in adding 1, 2, 3, etc. units to the index of the logarithm of a number, we multiply this number by 10, 100, 1000, etc., since this is to add the logarithm of 10, of 100, or 1000, etc.(219 and 227.)

2. On the contrary, if we subtract 1, 2, 3, etc. units from the index of a logarithm, we divide the corresponding number by 10, 100, 1000, etc.

239. This established, let it be required to find the logarithm of 357859.

I shall separate, by a comma, upon the right of this number, as many figures as are necessary, so that the remainder may be found in the table. Here, for instance, I shall separate two, this will give me 3578,59, which (28) is a hundred times less than the proposed number 357859.

I seek in the table the logarithm of 3578, and opposite 357, and under 8 I find that the decimal part of this logarithm is ,553640, to which, because the number consists of four figures, I prefix the index 3, and I have 3,553640; I take at the same time by the side of this logarithm,† the difference 122, between this logarithm and that of 3579; after which I make this rule of three: if, for 1, unit of difference between the two numbers 3579 and 3578, we have 122 difference between their logarithms, how much for 0,59 difference between the two numbers 3578,59 and 3578, shall we have of difference between their logarithms? That is to say, I seek the fourth term of a proportion, of which the three first are:

$$1 : 122 :: 0,59 :$$

This fourth term is 71,98, or 72 adding a unit to 71 because the decimals separated exceed 0,5. I add then 72 to the logarithm 3,553640 of 3578, and I have 3,553712 for the logarithm of 3578,59; all that is now required to have that of 357859, is to add two units to the index of the logarithm

† The differences are found by the side of the logarithms in the column marked Diff.

which we have found; we shall therefore have 5,55[illegible] for the logarithm sought, since 357239 is 100 times greater than 3572,39.

If the figures which we separate on the right be all ciphers, after having found in the table the logarithm of the part which remains on the left, all that is required is to add as many units to the index as we shall have separated ciphers.

240. If we require the logarithm of a number accompanied by decimals, we shall seek this logarithm, as if the proposed number had no comma: and having found it, whether immediately in the table, or by the method given, (233.) we shall subtract as many units from the index, as there are decimals in the proposed number, because having considered the number as if it had no comma, that is to say, as 10, 100, or 1000, etc. times greater than it is, we should bring it to its value by a suitable diminution of the index of its logarithm.

241. Lastly, if there be nothing but decimals in the proposed number, we shall still seek this number in the table as if it had no comma; and having taken the corresponding logarithm, we shall subtract it from as many units as there are decimals in this same number, and to the number we shall prefix the sign —; for example, to have the logarithm of 0,03, I seek that of 3 which is 0,477121: I subtract it from 2 units, and prefixing to the remainder the sign—, I have— 1,522879 for the logarithm of 0,03. In effect, 0,03 is nothing else than $\frac{3}{100}$: now, to have the logarithm of $\frac{3}{100}$, we must (235) subtract the logarithm of 3, from that of 100, and prefix to the remainder the sign—.

OF THE LOGARITHMS OF WHICH THE NUMBERS ARE NOT FOUND IN THE TABLE.

242. This research is no less necessary than the preceding. For example, in division, it rarely happens that the quotient is a whole number. Now, if we perform the operation by logarithms, we shall not find in the table the remaining logarithm, except when the quotient shall be a whole

number. There is an infinity of other cases of the same kind.

243. Let us propose to find the number answering to a proposed logarithm, whether it exceed the limits of the table, or whether it fall between the logarithms of the table.

We shall subtract from the index as many units as are necessary in order to find in the table the first figures of the proposed logarithm thus prepared. If all the figures be then found in the table, the number sought will be the same number that we find opposite to this logarithm in the table; but we must place as many ciphers on the right of it as we have taken units from the index.(238.)

For example, the logarithm 7,696007 is found (after having taken 4 units from its index,) to answer to the number 4966; I therefore conclude that the proposed logarithm 7,696007, answers to 49660000.

If we find in the table only the first figures of the logarithm, we operate as in the following example:

To find the number answering to the logarithm 5,243276, I take 2 units from its index; the logarithm 3,243276 which I then have, falls between the logarithms of 1750 and 1751: the number to which it answers is then 1750 and a fraction.

In order to have this fraction, I subtract from my logarithm 3,243276, the logarithm of 1750, and I have for the difference 238.

I take also in the table the difference 248 between the logarithms of 1751 and 1750; after which I make the following Rule of Three:

If 248, the difference between the logarithms of 1751 and 1750,

Answers to 1, unit of difference between these numbers,

To what difference of the numbers should 238 answer, which is the difference between my logarithm and that of 1750?

I find for the fourth term $\frac{238}{248}$, or $\frac{119}{124}$; therefore, the logarithm 3,243276 belongs to the number $1750\frac{119}{124}$, very nearly; consequently, the proposed logarithm, which belongs to a number 100 times greater,(238) has for its correspondent

number $175000\frac{11900}{124}$; that is to say, $175095\frac{120}{124}$, or, in reducing the fraction to decimals, it has for its correspondent number 175095,97.

244. If the proposed logarithm should fall between those of the table; there would be no unit to subtract from its index, and, consequently, no cipher to add at the end of the operation, which we should otherwise perform in the same manner.

245. But as the proportion that we employ in this method is not rigorously exact,† and as it only approaches near the truth, when the numbers sought are great; if the proposed logarithm falls below that of 1500, we must for greater exactness, add to its index as many units as we can without passing the bounds of the table; and having found the number which comes the nearest to it in the table, we shall separate on the right as many figures, by a comma, as we have added units to its index, which will, in general, suffice; but if we would have more decimals, we shall make the proportion as above,(243) and, reducing the fourth term to decimals, we shall place these on the right of those which we have already found.

For example, if we would have the number answering to the logarithm 0,543273; as this logarithm falls between 3 and 4, and as the number to which it belongs is, consequently, much under 1500, I seek this logarithm, with three units more to its index; that is to say, I seek 3,543273; I find that it falls between the logarithms of 3493 and 3494, whence I conclude that the number sought is 3,493, to within a thousandth. But if this approximation does not suffice, I shall take the difference between my logarithm and that of 3493, which is 74; I shall likewise take the difference 124 between the logarithms of 3494 and 3493, and I shall seek, reasoning as above, the fourth term of a proportion which commences with these three:

$$124 : 1 :: 74 :$$

† This proportion supposes that the differences of the logarithms are proportional to the differences of the numbers, which is never exactly true, but approaches near enough, when the numbers are somewhat great, and this is sufficient for all ordinary purposes.

This fourth term is worth in decimals 0,59; wherefore the number sought is 3,49359.

This second approximation is also limited, because the logarithms of the table being only exact to within about half a decimal unit of the sixth order, the differences are affected by this small error; but we can always carry the approximation with confidence as far as two decimals: the remark that we here make ought also to direct in the use that we have made above (239 and 243,) of the same proportion.

246. If we would have the fraction answering to a proposed negative logarithm, we must subtract this logarithm from 1, 2, 3, or 4, etc. units, according to the extent of the table; and having found the number which answers to the remaining logarithm, we shall separate upon the right, by a comma, as many ciphers as there were units in the number from which we subtracted the logarithm.

For example, to have the fraction answering to —1,532732, I subtract 1,532732 from 4, there remains 2,467268, which in the table is found to be between the logarithms of 293 and 294; I conclude thence that the fraction sought is between 0,0293 and 0,0294; that is to say, that it is 0,0293 to within a ten-thousandth. In effect, to subtract from 4, the proposed logarithm 1,532732 is (236) to multiply 10000 by the fraction to which this same proposed logarithm belongs, or which is the same thing, it is to multiply this fraction by 10000; therefore, the number which we find is 10000 times too great; we must therefore consider it as ten-thousandths.

All that we have said will have abundant application in the sequel.

Let us be satisfied for the present with giving an idea, by some examples, of the facility and promptitude with which calculations are performed by means of logarithms.

Example I.

What is the quotient of 17954 divided by 12836 to within a thousandth?

Logarithm of 17954	-	4,254161
Logarithm of 12836	-	4,108430
Remainder		0,145731

This remainder, sought in the table with an index of three units, answers to 1396; therefore (238) the quotient sought is 1,396.

Example II.

What is the cube root of 53 to within a thousandth?

The logarithm of 53 is	-	1,724276
Its third (230) is	- -	0,574759

This last, sought in the table with an index of three units, answers to 3756; therefore (238) the root sought is 3,756.

To judge of the advantage of logarithms, we have only to seek this root by the method given (156.) We must not on this account regard this last as useless; for it extends to an infinity of numbers to which logarithms would not attain, by reason of the limits of the tables.

Example III.

We would have, to within a hundredth, the fifth root of the cube of 5736.

We triple the logarithm 3,758609, of 5736; and we have 11,275827 for the logarithm of the cube of 5736. Taking the fifth of this last logarithm, we have 2,255165 for the logarithm of the fifth root of the cube of 5736. This logarithm, sought in the table with an index greater by two units, to have hundredths, answers between the numbers 17995 and 17996; the root sought is then 179,95 to within a hundredth.

Example IV.

Let it be required to find four geometrical mean proportionals between $2\frac{2}{3}$ and $5\frac{3}{4}$.

We must (215,) to have the ratio which should reign in the progression, divide $5\frac{3}{4}$ by $2\frac{2}{3}$, and extract the fifth root of the quotient.

By logarithms, this operation is very simple. I determine by the table the logarithm of $5\frac{3}{4}$, or $\frac{23}{4}$; it is 0,759668. I also determine the logarithm of $2\frac{2}{3}$; it is 0,425969. I then subtract (231) this logarithm from the first, and I have

0,333699; then taking (230) the fifth of this last, I have 0,066740 for the logarithm of the ratio sought. This logarithm, sought in the table with an index of 4 units, to have 4 decimals, answers to 11661, to within a unit; therefore, the ratio is 1,1661, to within a ten-thousandth. All that we require then, to have the mean proportionals, is to multiply the first term $2\frac{2}{3}$ by 1,1661, and this product by 1,1661, and so on.

But these operations may be performed much more promptly by the aid of logarithms, in adding successively to the logarithm 0,425969 of the first term $2\frac{2}{3}$, the logarithm 0,066740 of the ratio, its double, its triple, and its quadruple, so that we shall have 0,492709; 0,559449; 0,626189; and 0,692929 for the logarithms of the four mean proportionals required. And if we seek these logarithms in the table, with three units for their index, we find that these four mean proportionals are 3,109; 3,626; 4,228; and 4,931.

Remark.

When in an operation where we make use of logarithms, there are found some which should be subtracted, we can simplify the operation as follows:

When we have to subtract any number from another which is a unit, followed by as many ciphers as there are figures in the first, the operation is reduced to the writing of the difference between 9 and each of the figures of the proposed number, with the exception of the unit or last figure, for which we write the difference between 10 and this figure. For example, if I have 526927 to subtract from 1000000, I subtract successively the figures 5,2,6,9,2 from 9, and the last figure 7 I subtract from 10, and I have 473073 for the remainder.

This remainder is what is called the *arithmetical complement* of the proposed number.

The subtraction made in this manner being too simple to be considered an operation, it thence follows that when we shall have to form a result of the addition and subtraction of several numbers, we may always reduce the operation to addition. For example, if it be required to add the two num-

bers 672736, 426452, and to subtract from their sum the two numbers 432752, 18675, which requires two additions and a subtraction, I substitute for this operation the following:

	672736
	426452
Arithmetical complement of 432752	567248
Arithmetical complement of 18675	981325
Sum	2647761

that is to say, I add together the two numbers first proposed, and the arithmetical complements of the two last; the sum is 2647761. We must suppress the first figure 2, and the remaining figures 647661 are the required result.

The reason of this operation is easy to perceive, in observing that if instead of subtracting 432752 as we proposed, I add its arithmetical complement, that is to say, 1000000 minus 432752, I make at the same time the proposed subtraction and an augmentation of 1000000, that is to say, of a ten to the first figure of the result; therefore, for each arithmetical complement that I have introduced, I shall have a ten too much with regard to the first figure of the result.

The application of this to logarithms is evident.

Let it be required, for example, to divide 3760 by 79.

We should in this case subtract the logarithm of 79 from that of 3760.

Instead of this operation, I write log. 3760	3,575188
Arithmetical complement of log. 79	8,102373
Sum	11,677561

Therefore, 1,677561 is the logarithm of the quotient, and answers to 47,59 to within a hundredth.

Let us suppose, for second example, that it be required to multiply $\frac{675}{527}$ by $\frac{952}{377}$; we ought (106) to multiply 675 by 952, and 527 by 377; and then to divide the first product by the second.

By logarithms, we shall operate thus:

Logarithm of 675 - - -	2,829304
Logarithm of 952 - - -	2,978637
Arithmetical complement of the log. of 527	7,278189
Arith. comp. of log. of - - 377	7,423659
Sum	20,509789

The logarithm of the result is then 0,509789, which, sought with three units for its index, answers to 3,234: wherefore, 3,234 is the required product.

We can make use of the arithmetical complement, to put the logarithms of fractions under the form of those of whole numbers, and employ them in the same manner in the calculation; by this means we shall avoid the distinction of negative and positive logarithms. It will suffice to remember that the index of a logarithm of a fraction, properly so called, is too great by 10 units.

For example, to have the logarithm of $\frac{3}{4}$, which (96) is nothing else than 3 divided by 4; instead of subtracting the logarithm of 4 from that of 3; that is to say, instead of subtracting the logarithm of 3 from that of 4, and prefixing to the remainder the sign — (235;) to the logarithm of 3, I add the arithmetical complement of the logarithm of 4:

Logarithm of 3 - - -	0,477121
Arith. comp. of log. of 4 - -	9,297940
Sum	9,775061

This sum is the logarithm of $\frac{3}{4}$, the index of which is too great by 10 units. Now, it is not necessary actually to make the diminution; we can reject ten at the end of the operations in which we shall employ this logarithm.

The same rule applies also to decimal fractions; thus to have the logarithm of 0,575, which is nothing else than $\frac{575}{1000}$, to the logarithm of 575, I should add the arithmetical complement of the logarithm of 1000.

In thus employing the arithmetical complements, instead of the negative logarithms of fractions, it is not more difficult to find, in the table, the value of these same fractions. Since

when I know that a proposed logarithm is, or contains one or more arithmetical complements, I know that its index is too great by as many tens as there are arithmetical complements in its formation; therefore if it surpass this number of tens, it will be easy to diminish it, and to find the number to which this logarithm belongs, which will be a whole number or a whole number joined to a fraction.

But if the index be below the number of tens by which it is too great, it certainly belongs to a fraction which I shall find in this manner: I seek, by what has been said (24 and fol.) to what number the proposed logarithm answers; and when I have found it, I separate by a comma, as many tens of figures on the right, as there are tens too many in the index.

For example, if I had 8,732235 for the logarithm resulting from an operation in which there has entered an arithmetical complement, I see, since its index is below ten, that it belongs to a fraction. I seek first (242) to what number 8,732235 answers, considered as a whole number; I find that it answers to 539802500; separating 10 figures, I have 0,0539802500, for the very near value of the fraction which answers to the proposed logarithm.

But as it is rarely necessary to have these fractions to such a degree of precision, we shall abridge the operation in diminishing the index of the proposed logarithm, as much as is requisite to make it fall among those of the table, and taking simply the corresponding number, we shall separate as many figures less than the number prescribed by the preceding rule, as we have taken units from the index. Thus, in the present case, I diminish the index by 5 units, and having found that the corresponding number is 5398, I separate only five figures, and I have 0,05398.

In the raising of powers we must observe, that in multiplying (226) the logarithm by the number which is the index of the power, we shall find that we also multiply that by which the index is too great. Thus, in raising to the cube for example, if there enters an arithmetical complement, that is to say, if the index be too great by 10 units, that of

the logarithm of the cube will be too great by 30 units, and so on for the others. It will then be easy to bring it to its just value.

In the extraction of the roots, to avoid all error, when arithmetical complements enter into the logarithms of which we make use, we should be careful to add to the index or to subtract from it as many tens as are necessary, to the end that the number by which it is too great may be precisely as many tens as there are units in the index of the root: and having, conformably to the ordinary rule, divided by the index of the root, the index of the logarithm will be precisely 10 units too great.

For example, if we would have the cube root of $\frac{276}{547}$; to the logarithm of 276, I add the arithmetical complement of the logarithm of 547.

Logarithm of 276	- - -	2,440909
Arith. comp. of log. 547	- - -	7,262013
	Sum -	9,702922
to the index of which I add	- -	20
		29,702922

in order that it may be too great by 3 tens, and I have 29,702922, the third of which, 9,900974, is the logarithm of the cube root required. Therefore, conformably to what has been said above, I find that this cube root is 0,7961 to within a ten thousandth.

The arithmetical complements are chiefly useful in the calculations of trigonometry, consequently in many operations of pilotage and surveying which we would perform to a certain degree of exactness.

*USE OF THE ARITHMETICAL PROGRESSION, (Page 160.)

PROBLEM I.

A certain person, indebted to another, acquitted himself in several payments, each of which exceeded the preceding by $4; the first payment was $3, and the last $91. We demand, 1. What was the number of payments? 2. The sum due?

Problem V.

$1081 was paid in several payments, the difference of each payment was $4, and the first payment was $3. We demand, 1. What was the number of payments? 2. The last payment?

1. Since we know the sum of the terms $1081(s,)$ the difference $4(d,)$ the first term $3(a,)$ and that

$$n=\frac{1}{2}-\frac{a}{d}+\sqrt[2]{\frac{2s}{d}+\frac{1}{4}-\frac{a}{d}+\frac{a^2}{d^2}},$$

we shall have

$$n=\frac{1}{2}-\frac{3}{4}+\sqrt[2]{\frac{2\times1081}{4}+\frac{1}{4}-\frac{3}{4}+\frac{3\times3}{4\times4}}=23.$$

2. Since we know the first term $3(a,)$ the difference $4(d,)$ the number of terms $23(n,)$ and that $\omega=a+dn-d$, we shall have

$$\omega=3+4\times23-4=91.$$

The number of payments was 23, and the last payment $91.

Problem VI.

A certain person paid $1081, in twenty-three payments, which surpassed each other equally; the first was $3. We demand, 1. What was the last payment? 2. What was the difference of the payments?

1. Since the sum of the terms $s=1081$; the number of terms $n=23$; the first term $a=3$, and since $\omega=\frac{2s}{n}-a$, we shall have

$$\omega=\frac{2\times1081}{23}-3=91 \text{ dollars the last payment.}$$

2. Since the last term $\omega=91$; the first term $a=31$; the number of the terms $n=23$, and $d=\frac{\omega-a}{n-1}$ we shall have

$$d=\frac{91-3}{23-1}=4 \text{ dollars the difference of the payments.}$$

Problem VII.

We have paid an unknown sum in twenty-three payments, which surpassed each other by $4; the last payment was $91. We demand, 1st. What was the first? 2d. What was the sum discharged?

1. Since the number of terms $n=23$; the difference $d=4$; the last term $\omega=91$, and since $a=\omega-dn+d$, we shall have

$$a=91-4\times23+4=3 \text{ dollars the first payment.}$$

2. Since we know that the first term $a=3$; that the number of the terms $n=23$; that the last term $\omega=91$, and that $s=\frac{an+\omega n}{2}$, we shall have

$$s=\frac{3\times23+91\times23}{2}=1081 \text{ dollars, the sum discharged.}$$

Problem VIII.

We have paid $1081 in several payments, which surpassed each other by $4; the last payment was $91. We demand, 1. What was the number of payments? 2. What was the first payment?

Since we know the sum of the term s which is equal to 1081; that the difference $d=4$; that the last term $\omega=91$, and that $n=\frac{1}{2}+\frac{\omega}{d}+\sqrt{-\frac{2s}{d}+\frac{1}{4}+\frac{\omega}{d}+\frac{\omega^2}{d^2}}$ we shall have

$$n=\tfrac{1}{2}+\tfrac{91}{4}+\sqrt{\left(\frac{-2\times1081}{4}+\tfrac{1}{4}+\tfrac{91}{4}\times\frac{91^2}{4^2}\right)}=23.$$

Problem IX.

We have paid $1081 in twenty-three payments, which surpassed each other equally; the last was $91. We demand, 1. What was the first payment? 2. What was the difference of the payments?

1. Since we know the sum of the terms s which is equal 1081: that the number n of the terms is equal to 23; that

the last term ω is equal to 91, and that $a=\frac{2s}{n}-\omega$, we shall have $a=\frac{2\times1081}{23}-91=3$ dollars the first payment.

2d. Since we know the number of terms n which is equal to 23; that the first term a is equal to 3; that the last term ω is equal to 91, and that $d=\frac{\omega-a}{n-1}$, we shall have $d=\frac{91-3}{23-1}=4$ dollars the difference of the payments.

Problem X.

A person paid $1081 in twenty-three payments, which surpassed each other by $4. We demand, 1. What was the first term? 2. What was the last?

1. Since we know the sum of the terms s which is equal to 1081; that the number of the terms n is equal to 23; that the difference d is equal to 4, and that $a=\frac{s}{n}-\frac{dn-d}{2}$, we shall have $a=\frac{1081}{23}-\frac{4\times23-4}{2}=3$ the first term.

2. Since we know the number of the terms n which is equal to 23; the difference d which is equal to 4; the first term a equal to 3, and that $\omega=a+dn-d$, we shall have

$$\omega=3+4\times23-4=91 \text{ the last term.}$$

USE OF THE GEOMETRICAL PROGRESSION FOR THE CALCULATION OF COMPOUND INTEREST, (Page 166.)

In every rule of compound interest, the sums due, at the end of the several years, form a geometrical progression, the first term of which is the capital, the second the capital multiplied by a unit augmented by the interest, the third is equal to the second multiplied by a unit augmented by the interest and so on.

Let the capital be, for example, $1200, and the interest at ten per cent., or, which is the same thing, $\frac{1}{10}$ per dollar. It is evident that the sum due at the end of the first year will be equal to once the capital $1200, plus 1200 times $\frac{1}{10}$ of a dollar; that is to say,

$$1200\times(1+\tfrac{1}{10}),$$

that is to say,

$$1200\times\tfrac{11}{10};$$

that the sum due at the end of the second year will equal

$$1200\times\tfrac{11}{10}\times\tfrac{11}{10};$$

that is to say,

$$1200\times(\tfrac{11}{10})^2;$$

that the sum due at the end of the third year will equal

$$1200\times(\tfrac{11}{10})^3;$$

and so on in succession.

Now, it is evident that the numbers 1200, $1200\times\frac{11}{10}$, $1200\times(\frac{11}{10})^2$, $1200\times(\frac{11}{10})^3$, etc. form a geometrical progression, of which the first term a equals 1200, the quotient q equals $\frac{11}{10}$, and of which any term ω equals $1200\times(\frac{11}{10})^{n-1}$.

Therefore, calling a the first term, r the interest of a dollar, the quotient will equal $1+r$, and we shall have

$$\because a : a\times(1+r) : a\times(1+r)^2 : a\times(1+r)^3 : \ldots.$$

$$\omega=1200\times(1+r)^{n-1}.$$

What I have said of compound interest applies, word for word, to the increase of population.

To make use of the table, (page 166,) for the resolving of all the problems relative to the rules of compound interest, and to the increase of population, it will suffice then to substitute $1+r$ for q. Thus, instead of the formulæ

$$q=\sqrt[n-1]{\frac{\omega}{a}},\quad n=1+\frac{L\omega-La}{Lq},$$

we shall have, first,

$$1+r=\sqrt[n-1]{\frac{\omega}{a}},$$

that is to say,

$$r=-1+\sqrt[n-1]{\frac{\omega}{a}}=-1+\frac{L\omega-La}{Ln-1}.$$

we shall have, secondly,

$$n=1+\frac{L\omega-La}{L(1+r)}.$$

Problem I.

We have placed at interest \$1000000, at the rate of ten per cent. compound interest. What sum shall we touch at the end of five years and a half?

The sum that we shall touch at the end of the fifth year will equal aq^5 plus the half of the interest of aq^5 during a year; that is to say,

$$aq^5 \times \tfrac{1}{10} \times \tfrac{1}{2},$$

therefore, the sum that we shall touch at the end of five years and a half will equal

$$aq^5 + aq^5 \times \tfrac{1}{10} \times \tfrac{1}{2} = 1000000(\tfrac{11}{10})^5 + 1000000(\tfrac{11}{10})^5 \times \tfrac{1}{10} \times \tfrac{1}{2} =$$
$$\$1691035\tfrac{1}{2}.$$

Problem II.

We have touched \$1610510 for a capital, placed five years since; the compound interest being one-tenth per dollar, what was this capital?

Since we know the last term 1610510 (ω), the number of terms 6 (n), the quotient $\frac{11}{10}$ (q), and that $a = \frac{\omega}{q^{n-1}}$, we shall have

$$a = \frac{1610510}{(\frac{11}{10})^{6-1}} = \frac{1610510}{\frac{11^5}{10^5}} = \frac{1610510 \times 10^5}{11^5} = \frac{1610510 \times 100000}{161051}$$
$$= 1000000 \text{ dollars, the required capital.}$$

Problem III.

We have touched \$1691035½ for a capital, at interest for five years and a half, the compound interest being at ten per cent. What was this capital?

Let a be the capital, the sum touched at the end of the fifth year will be $a \times (\frac{11}{10})^5$, the sum touched for the six remaining months will equal the half of the interest of $a(\frac{11}{10})^5$ during a year; that is to say,

$$a(\tfrac{11}{10})^5 \times \tfrac{1}{10} \times \tfrac{1}{2},$$

that is to say,

$$a(\tfrac{11}{10})^5 \times \tfrac{1}{20}.$$

We shall then have

$$a(\tfrac{11}{10})^5 + a(\tfrac{11}{10})^5 \times \tfrac{1}{20} = \omega,$$

that is to say,

$$a \times [(\tfrac{11}{10})^5 + (\tfrac{11}{10})^5 \times \tfrac{1}{20}] = \omega;$$

therefore,

$$a=\frac{\omega}{(\frac{11}{10})^5+(\frac{11}{10})^5\times\frac{1}{20}}=\frac{1691035+\frac{1}{2}}{(\frac{11}{10})^5+(\frac{11}{10})^5\times\frac{1}{20}}$$

$$=\frac{1691035+\frac{1}{2}}{\frac{161051}{100000}+\frac{161051}{100000}\times\frac{1}{20}}=\frac{1691035+\frac{1}{2}}{\frac{3221020+161051}{2000000}}$$

$$=\frac{(1691035+\frac{1}{2})2000000}{3382071}=\frac{3382071\times1000000}{3382071}=1000000$$

dollars the required capital.

Problem IV.

We have put to interest \$1000000, and have touched \$1610510 at the end of five years. What was the compound interest?

Since we know the first term 1000000(a,) the last term 1610510(ω,) the number of the terms 6(n,) and that

$$q=\sqrt[n-1]{\frac{\omega}{a}}$$

we shall have $$q=\sqrt[5]{\frac{1610510}{1000000}}=\frac{11}{10};$$

therefore the interest was a tenth per dollar, or ten per 100.

Problem V.

Having placed at interest \$1000000 at ten per cent. we have touched \$1691035$\frac{1}{2}$. We demand what time the sum was at interest?

Since we know the first term 1000000(a,) the last term 1691035$\frac{1}{2}$(ω,) and the interest which is a tenth per dollar; the quotient q will then be $\frac{11}{10}$. But the number of the terms

$$n=1+\frac{L\omega-La}{Lq};$$

therefore $$n=1+\frac{L(1691035+\frac{1}{2})-L1000000}{L11-L10}=6\frac{21188}{41393}.$$

But the sum touched at the end of the fifth year is equal to $1000000(\frac{11}{10})^5$,

that is to say, 161050;

therefore $1691035 \div \frac{1}{2} - 1610510$,

that is to say, $80525\frac{1}{2}$ is the interest of 1610510 during a time p of the sixth year. But this interest is equal to

$$1610510 \times \tfrac{1}{18} \times p,$$

that is to say, $161051 \times p$;

therefore $161051 \times p = 80525 + \frac{1}{2}$;

therefore $p = \dfrac{80525 + \frac{1}{2}}{161051} = \frac{1}{2}$. The sum was therefore at interest $5\frac{1}{2}$ years.

I suppose, in the three following problems, that the population in France augments by a hundredth per year, that is to say, that a family composed of one hundred persons to-day will contain a hundred and one at the end of one year. This is not a vague supposition; the last lists prove that the population of France, for the last twenty-five years, in spite of the cruel wars, is considerably augmented. This increase has two principal causes: the suppression of monasteries and the discovery of vaccination.

If a long peace reigned in France, if vaccination were generally practised, and finally, if celibacy had ceased for ever to be placed among the number of the virtues, it is beyond a doubt that there would arrive an epoch when France would no longer be able to nourish its inhabitants; and this is proved by the three following problems.

Problem VI.

Suppose that there are 30000000 of inhabitants in France and that the population augments by a hundredth per year. According to these two suppositions, how many inhabitants will there be in France at the end of a hundred and one years?

Agreeably to the formula $\omega = aq^{n-1}$; we shall have

$$\omega = 30000000 \times \left(\tfrac{101}{100}\right)^{100}.$$

Operating by logarithms, we shall find for the logarithm

(101.) $2{,}00432 - 2.00000$.

that is to say, $0{,}00432$.

and for the log. of $(\frac{101}{100})^{100}$, 0,43200,
which corresponds to the number $\frac{2704}{1000}$.

Therefore,

$$\omega = 30000000 \times \frac{2704}{1000} = 30000 \times 2704 = 81120000.$$

Therefore, at the end of a hundred and one years, France will contain 81120000 inhabitants.

Problem VII.

Suppose that in France there are 30000000 of inhabitants, that the population augments by a hundredth per year. In how many years will the population amount to 81120000 inhabitants?

According to the formula

$$n = 1 + \frac{L\omega - La}{L(1+r)},$$

we shall have

$$n = 1 + \frac{L.81120000 - L.30000000}{L.101 - L.100}$$

since $L.81120000 - L.30000000 = 0,43200$,
and since $L.101 - L.100 = 0,00432$.

we shall have $n = 1 + \frac{0,43200}{0,00432} = 1 + 100 = 101$.

Therefore in a hundred and one years the population of France will be 81120000.

Problem VIII.

There are 30000000 inhabitants in France; in what proportion must the population augment so that there may be 81120000 at the end of a hundred and one years?

According to the formula,

$$r = -1 + \sqrt[n-1]{\frac{\omega}{a}} = -1 + \frac{L.\omega - L.a}{n-1} =$$

$$\frac{L.81120000 - L.30000000}{10} = -1 + \frac{0,43200}{100} = -1 + 0,00432;$$

now the logarithm 0,00432 answers to the number $\frac{101}{100}$;
therefore $r = -1 + \frac{101}{100} = \frac{1}{100}$.

The population must therefore augment by a hundredth yearly.

OF ANNUITIES.

We call annuities equal sums which are paid each year to extinguish, in a given time, a capital and its compound interest.

A person has borrowed $12000 for five years, at the rate of ten per cent.; he would acquit himself in five equal payments made at the end of each year. What is the sum that he will have to pay at the end of each year?

It is evident that, if there were no reimbursements, he would be indebted $12000\times(\frac{11}{10})^5$ at the end of the fifth year.

But the borrower remits a sum b, at the end of the first year, without which b would have become $b\times(\frac{11}{10})^4$; he remits b at the end of the second year, without which b would have become $b\times(\frac{11}{10})^3$; he remits b at the end of the third year, without which b would have become $b\times(\frac{11}{10})^2$; he remits b at the end of the fourth year, without which b would have become $b\times\frac{11}{10}$, and lastly, he remits b at the end of the fifth year: therefore

$$b\times(\tfrac{11}{10})^4+b\times(\tfrac{11}{10})^3+b\times(\tfrac{11}{10})^2+b\times(\tfrac{11}{10})+b=12000(\tfrac{11}{10})^5.$$

But the numbers

$$b\times(\tfrac{11}{10})^4+b\times(\tfrac{11}{10})^3+b\times(\tfrac{11}{10})^2+b\times\tfrac{11}{10}+b$$

form a geometrical progression, the sum of which is equal to

$$\frac{b\times(\frac{11}{10})^4\times\frac{11}{10}-b}{\frac{11}{10}-1},$$

that is to say $\dfrac{b\times(\frac{11}{10})^5-b}{\frac{1}{10}}$.

therefore $\dfrac{b\times(\frac{11}{10})^5-b}{\frac{1}{10}}=12000\times(\frac{11}{10})^5$

but $\dfrac{b\times(\frac{11}{10})^5-b}{\frac{1}{10}}=\dfrac{b\times[(\frac{11}{10})^5-1]}{\frac{1}{10}}$

therefore $\dfrac{b\times[(\frac{11}{10})^5-1]}{\frac{1}{10}}=12000\times(\frac{11}{10})^5$

therefore $b=\dfrac{12000\times(\frac{11}{10})^5\times\frac{1}{10}}{(\frac{11}{10})^5-1}=3165\frac{34738}{61051}$ Ans.

It will be the same whatever be the sum lent, the interest and the number of years.

Calling a the sum lent, n the number of years, r the interest, and b the sum which must be remitted at the end of each year, we shall have

$$b=\frac{a\times r\times(1+r)^n}{(+r)^n-1},$$

from which we deduce:

$$a=\frac{b\times(1+r)^n-b}{r\times(1+r)^n},\quad n=\frac{L[b-L(b-a\times r)]}{L(1+r)}.$$

With these three formulæ we can resolve all the cases that annuities present.

Let it be proposed, for example, to find what sum we must give yearly, to extinguish, in three years, a debt of \$500, the interest being at ten per cent.

Substituting these numbers in the first formula, we shall find $b=\$201+\frac{19}{331}$. Therefore, $\$201\frac{19}{331}$, or \$201,057+ is the yearly payment.

If the interest were at 7 per cent., we should find $b=\$190,526-$. This last answer being somewhat too great, we annex the sign —.

CONJOINED PROPORTION.

This rule is thus named, because it serves to find the relation of one number to another by means of several proportions.

Let a be the pound of Venice, b that of Lyons, c that of Rouen, d that of Toulouse, e that of Geneva, and let

$$\begin{aligned} 100a &= 70b \\ 120b &= 100c \\ 80c &= 100d \\ 100d &= 74e; \end{aligned}$$

It is required to find the relation of the pound of Venice a with the pound of Geneva e. The above equations furnish the following proportions:

$$a : b :: 70 : 100$$
$$b : c :: 100 : 120$$
$$c : d :: 100 : 80$$
$$d : e :: 74 : 100$$

This is evident since the products of the extremes are equal to the products of their means. But when we have several proportions, the product of the first antecedents of these proportions is to the product of their first consequents as the product of their second antecedents is to the product of their second consequents; therefore

$$a \times b \times c \times d : b \times c \times d \times e,$$

that is to say,

$a : e :: 70 \times 100 \times 100 \times 74 : 100 \times 120 \times 80 \times 100 :: 259 : 480$; therefore

$$a : e : 259 : 480,$$

therefore

$$a = \frac{259}{480} \times e;$$

The pound of Venice is therefore equal to the two hundred and fifty-nine four hundred and eightieths of that of Geneva.

RULE OF A FALSE POSITION.

This rule serves to find one or more unknown numbers according to the conditions given.

Problem I.

To find a number of which the half, the third, and the two-fifths make 148.

We could take, to resolve this problem, any number whatever; but to facilitate the calculation, we choose a number which is divisible by 2, by 3, and by 5. The least common multiple of these three numbers is 30. In all cases, the product of the numbers 2, 3, 5 will necessarily be divisible by 2, 3, 5. It will be the same for any other numbers.

But the half of 30 is 15, the third 10, and the two-fifths 12; and $15+10+12=37$.

Hence we make the following proportion:

$$37 : 30 :: 148 : x = 120.$$

See here the demonstration of this rule:

Let x be the number sought, we shall have

$$\frac{30}{2} : \frac{x}{2} : : \frac{30}{3} : \frac{x}{3} : : \frac{2}{5} \times 30 : \frac{2x}{5};$$

therefore

$$\frac{30}{2} + \frac{30}{3} + \frac{2}{5} \times 30 : \frac{x}{2} + \frac{x}{3} + \frac{2x}{5} : : \frac{30}{2} : \frac{x}{2} : : 30 : x. \quad (186)$$

But $$\frac{x}{2} + \frac{x}{3} + \frac{2x}{5}$$

should be equal to 148; therefore

$$\tfrac{30}{2} + \tfrac{30}{3} + \tfrac{2}{5} \times 30 : 148 : : 30 : a = 120.$$

therefore $$\tfrac{120}{2} + \tfrac{120}{3} + \tfrac{2}{5} \times 120 = 148.$$

The demonstration would be the same for all analogous questions.

Problem II.

A person bequeaths his estate to three of his friends; he gives the third of it to the first; the two-fifths to the second; and the $32000 which remained to the third. We demand what was the sum total of the estate of the testator, and the parts of the two first friends.

Let x be the estate of the testator, and let us choose the number 15, which is divisible by 3 and by 5. Taking from 15 its third and its two-fifths, we shall have 4, and we shall make the following proportion:

$$4 : 15 : : 3200 : x = 120000.$$

The parts of the two first friends will be $40000, and $48000, the sum of which $88000 taken from $120000 gives $32000 for the third friend's share.

The preceding demonstration applies equally to this question. In effect, let x be the estate of the testator, and choosing the number 15, we shall have

$$\frac{15}{3} : \frac{x}{3} : : \frac{2}{5} \times 15 : \frac{2x}{5} : : 15 - \frac{15}{3} - \frac{2}{5} \times 15 : x - \frac{x}{3} - \frac{2x}{5},$$

that is to say,

$$15 : x : : 4 : \frac{4x}{15} : : 4 : 32000;$$

therefore

$$15 : \frac{x}{3}+\frac{2x}{5}+3200 : : 4 : 32000,$$

from whence we deduce

$$x=120000.$$

Problem III.

To divide 7800 into three parts, so that the second may be the double of the first, and that the third may be equal to both the others.

Let us choose a unit for the first. The first being 1, the second will be 2, and the third 3; let x be the first part, the second will be $2x$, and the third $3x$; we shall then have

$$1 : x : : 2 : 2x : : 3 : 3x.$$

Then $\quad 6 : 6x : : 1 : x;$

But $\quad 6x=7800.$

Therefore $\quad 6 : 7800 : : 1 : x=1300$ the first part.

The second part will therefore be 2600, and the third 3900.

RULE OF TWO FALSE POSITIONS.

This rule is thus called. because that by the aid of two numbers taken by chance, and which we call false, we discover the number sought.

We must proceed as follows:

Let us take a number by chance, and apply this number to the proposed question, as if it were the number sought.

If at the end, this number does not answer to the conditions of the question, we must write this number with the error which will be in plus or minus.

We afterward take another number, and we operate in the same manner; we write this number under the first with the error in plus or minus.

We multiply the first supposed number by the second error; we afterward multiply the second supposed number by the first error.

This done, if the two errors have the same signs, that is to say, if they both have the sign *plus*, or the sign *minus*, we di-

vide the difference of the products by the difference of the errors. The quotient will be the number sought.

If the errors have different signs, that is to say, if the one be in plus and the other in minus, we must divide the sum of the products by the sum of the errors, and the quotient will be the number sought.

Problem I.

A person bequeaths $100 to three persons. The second is to have double the part of the first minus $8, and the third triple that of the first minus $15. What must each person have?

Let us suppose, 1. that the part of the first is $15, that of the second will be 22, and that of the third 30; the sum of these three parts is 67. But this sum should be equal to 100; there is then an error of 33 in minus; by the side of the first supposition, I write the first error with the sign minus, thus, 15—33.

Let us suppose, 2. that the part of the first is $18, the part of the second will be 28, and that of the third will be 39, the sum of which is 85. But this sum should equal 100; there is therefore an error of 15 in minus; by the side of the second supposition, I write the second error with the sign minus, thus 18—15.

First supposition	15—33.
Second supposition	18—15.

The first supposition which is 15 multiplied by the second error 15, produces 225; the second supposition which is 18 multiplied by the first error 33, produces 594: the errors having the same signs, I divide 369, which is the difference of the products, by 18, which is the difference of the errors; because the errors have like signs, the quotient $20+\frac{1}{2}$ will be the part of the first, 33 the part of the second, and $46+\frac{1}{2}$ the part of the third, and the sum of these parts is equal to 100, therefore $20+\frac{1}{2}$ is the number sought.

The same question with different suppositions.

First Supposition. Let the part of the first be 30, the part of the second will be 52, and the part of the third 75; the

sum of these parts being 157, the first error will be 57 in plus; I write $30+57$.

Second Supposition. Let the part of the first be 15, the part of the second will be 22, and the part of the third 30; the sum of these three parts is 67; the second error is therefore 33 in minus: I write $15-33$.

First supposition	$30+57$.
Second supposition	$15-33$.

We shall have

$$30\times33=990, \text{ and } 15\times57=855.$$

The errors having different signs, I divide 1845, which is the sum of the products, by 90, which is the sum of the errors, and the quotient is $20+\frac{1}{2}$ as above.

Problem II.

The sum of the ages of three persons is 148 years; the second is four years older than the first, and the age of the third is equal to the sum of the ages of the two others. What is the age of each?

First supposition. Let the age of the first be 20, the age of the second will be 24, and the age of the third 44; the sum of these three ages is 88; and the error being 60 in minus, I write $20-60$.

Second supposition. Let the age of the first be 24 years, the age of the second will be 28 years, and the age of the third 52. The sum of these three ages is equal to 104; the error being 44 in minus, I write $24-44$.

First supposition	$20-60$.
Second supposition	$24-44$.

We shall have

$$20\times44=880, \text{ and } 24\times60=1440.$$

The errors having the same signs, I divide 560, which is the difference of the products, by 16, which is the difference of the errors; the quotient is 35. The age of the first is therefore 35, that of the second 39, and that of the third 74; and the sum of these three ages is equal to 148 years.

Problem III.

A general, after a battle, reviews his army; a third has perished, a fourth has been made prisoner, a fifth has fled; there remain only 13000 men. What was the number of soldiers before the battle?

First supposition. Let 120000 be the number of the soldiers before the battle. We shall have 40000 killed, 30000 made prisoners, 24000 fled.

$$40000+30000+24000+13000=107000.$$

But 107000 should be equal to 120000; the error is then 13000 in minus. I write 120000—13000.

Second supposition. Let 360000 be the number of the soldiers; we shall have 120000 killed, 90000 prisoners, 72000 fled. But $120000+90000+72000+13000=295000$; the error is then 65000 in minus. I write 360000—65000.

First supposition 120000—13000.
Second supposition 360000—65000.

I operate as above and I find 60000 soldiers, which answers to the conditions of the question.

Problem IV.

A farmer demanded of another how many sheep he had; the other answered, if I had half as many, a fourth, the two-thirds, and 5, in addition to what I now have, I should then have 150. How many had he?

1. Let 12 be what he had, to which adding its half 6, its fourth 3, its two-thirds 8, and 5, I find 34 for the sum, the difference of which with 150 is 116, in minus. We then have 12—116.

2. Let 24 be the number, to which adding its half 12, its fourth 6, its two-thirds 16, and 5 more, I find 63 for the sum, the difference with 150 is 87, in minus; we shall then have

12—116
24—87.

I divide the difference of the products 12×87, and 24×116, that is to say, the difference of the products 1044 and 2784,

which is 1740, by 29, the difference of the errors, and I find 60 for the quotient or answer.

Proof.—$60+\frac{60}{2}+\frac{60}{4}+\frac{2}{3}\times 60+5=150$.

Problem V.

Alexander and the philosopher Calisthenes reasoned together upon their ages, and those of Clitus and Ephestion, both friends of the conqueror. I am two years older than Ephestion, said Alexander; the age of Clitus is equal to both our ages and four years more, the sum of our three ages is 96. We demand the age of Alexander, of Clitus, and of Ephestion.

Let us suppose, first, that Alexander was 28 years, Ephestion would be 26, and Clitus 58; the sum of these is 112; but it should be equal to 96. The error is then 16 years in plus.

$$28+16.$$

Let us suppose, secondly, that Alexander was 20 years old, Ephestion would be 18, and Clitus 42; the sum of these is 80; but it should be 96. The error is then 16 in minus. We shall then have

$$28+16,$$
$$20-16.$$

I divide the sum of the products 28×16, 20×16, that is to say, 768, by the sum of the errors, which is 32; I find for the quotient 24; which number fulfils the conditions of the question.

RULE OF CHANGE.

The change is the profit that the bankers and negotiators make of their money; that is to say, that they gain as much as their money would gain them if it were placed at interest.

The difficulty of transporting money from one place to another, as well on account of its weight as the dangers incurred upon the roads, has given rise to the establishment of places which are called places of change: we find such in all great cities. By this means, we can transport any sum of money that we would, by a letter of change from a banker

or negotiator, in paying the sum that we would send and the change of that sum.

For the same reasons, travellers, particularly when they go into very distant countries, are very careful to furnish themselves with letters of change, for the principal cities which are found in their route.

Example I.

A person, going from Paris to Marseilles, requires a banker to enable him to touch 3000 francs neat, in this last city. We demand how much he should give to the banker for the change of 3000 fr., this change being at 3 per cent.

We shall say

$$100 : 3 :: 3000 : x = 90.$$

Therefore, in paying 3090 francs, the banker will furnish him with the letter of change which he requires.

Example II.

The same person remits 3000 fr. to a banker, the discount being always at 3 per 100. How much will he touch?

We shall say, if 103 gives 100, how much will 3000 fr.? that is to say,

$$103 : 100 :: 3000 : x = 2912 + \tfrac{64}{103} \text{ fr. Ans.}$$

RULE OF DISCOUNT.

Discount is a sum to deduct from the amount of a bill or note for a given time, the value of which we would have before its expiration; that is, before it becomes due. It is also a sum that bankers pay upon the value of a bill, payable in a given time, and the interest of which is stipulated at so much per year, or per month, according to the agreement between the borrower and the lender.

Discount is taken in several ways; when, for example, for the sum of $100 lent, a bill is made of $105, payable in a year; the discount is said to be within. But when from the sum lent, the bill of which should be paid in a year, the banker commences by deducting the interest agreed on, we say then that the discount is without. It is evident, that in

this way the banker gains not only the interest of the m which he lends, but also the interest of the interest.

Problem I.

To find the discount of the sum of $800, at 6 per 10 the discount being within, we shall say

$$106 : 6 :: 800 : x = \$45,28\tfrac{14}{53}. \text{ Ans.}$$

Problem II.

To find the discount of $800, at 6 per 100, the discou being without.

We shall say, if $100, interest and principal, give 6, ho much will $800 give? that is to say,

$$100 : 6 :: 800 : x = \$48. \text{ Ans.}$$

When we would discount a bill before its expiration, w proceed thus:

Problem III.

A person has given a bill of $15640, payable at 365 days; he consents to pay the discount at 6 per 100, with the powe of diminishing the discount for the time that he may be abl to pay before the expiration of the bill; he comes to acqu himself at the end of 240 days. We demand what he shoul pay.

We shall say, 365 days are to 240 days as $6 are to x.

$$x = 3 + \tfrac{69}{73};$$

that is to say, that the interest at the rate of 6 per 100 for year, is reduced to 3\tfrac{69}{73}$, during 240 days. We shall the say

$$106 : 103\tfrac{69}{73} :: 15640 : x = \$15336,82 \text{ cts. } \tfrac{9}{100}. \text{ Ans.}$$

RULE OF ALLIGATION OF THE FIRST KIND.

This rule serves to make known the price or the weig of several merchandises or metals mixed together; the pric or the weight is called the mean price or weight.

Problem.

A wine merchant has mixed together wines of differe prices, viz. 15 gallons of wine at 12*s.*, 96 at 15*s.*, and 146 11*s.* 6*d.*

The price of the mixture of these different wines is 3299*s*. and the number of gallons is 257.

Dividing 3299*s*. by 257 gallons, we shall find 12*s*. $10\frac{19}{257}$*d*., which will be the price of a gallon of this mixture.

RULE OF ALLIGATION OF THE SECOND KIND.

Problem I.

A merchant has wine at 21*s*., at 22*s*., at 29*s*., and at 30*s*. per gallon; he would make a mixture worth 25*s*. per gallon. How much of each kind must he take?

30*s*	- - - - -	2 gallons.
29	- - - - -	4
	25*s*.	
23	- - - - -	5
21	- - - - -	4
	Sum	15 gallons.

I write the different prices of the gallon of wine under each other in the order of their value, the highest price being the first. I compare, first, the highest price 30*s*. with the mean price 25*s*., and I write the difference 5 opposite to 23, which is the first price inferiour to 25.

I compare, secondly, the second price 29, superiour to 25, with 25, and I write the difference 4 opposite to 21, which is the lowest price.

Afterward, ascending, I compare, thirdly, the lowest price 21 with 25; and I write the difference 4 opposite 29, which is the second price superiour to 25.

I compare, 4thly, the first price 23, which is inferiour to 25, with 25, and I write the difference 2 opposite the highest price; this done, I add the differences 2, 4, 5, 4; their sum gives 15 gallons.

Therefore, to have a mixture the price of which should be 25*s*., we must take 2 gallons at 30*s*., 4 at 29*s*., 5 at 23*s*., and 4 at 21*s*.

In effect, 2 gallons at 30*s*. give 60*s*., 4 at 29*s*. give 116*s*., 5 at 23*s*. give 115*s*., and 4 at 21*s*. give 84*s*.

I add the numbers 60, 116, 115, and 84, and I divide the sum 375 by 15, which is the number of gallons of the mixture. I find 25 for the quotient; but 15 gallons at 25*s*. make 375*s*., that is to say 60*s*., 116*s*., 115*s*., and 84*s*.; therefore 25*s*. is the price of a gallon of the mixture.

Suppose at present that we would fill a barrel which contains 35 gallons with wine, we would know how much wine we must take at 30*s*., at 29*s*., at 23*s*., and at 21*s*.

If to make a mixture of 15 gallons of wine, we must take 2 gallons at 30*s*., how many must we take to make a mixture of 35 gallons, that is to say,

gal.		*gal.*		*gal.*		
15	:	2	: :	35	:	$4+\frac{2}{3}$
15	:	4	: :	35	:	$9+\frac{1}{3}$
15	:	5	: :	35	:	$11+\frac{2}{3}$
15	:	4	: :	35	:	$9+\frac{1}{3}$
				Sum		35 gallons.

We should prove as above.

Problem II.

We have wheat at 30*s*. per bushel,
Rye at - 24*s*.
Barley at - 20*s*.

We would make a mixture of these three sorts of grain, so that the price of a bushel of this mixture may be 22*s*.

Place in the order of their value the different prices of these kinds of grain under each other, the highest price being the first, and write on the right the price of the mixture.

Wheat	30*s*.		2
Rye	24*s*.	22*s*.	2
Barley	20*s*.		8+2
		Sum	14 bushels.

I compare, 1st, the highest price 30 with the price of the mixture 22, and I write the difference 8 opposite the least price 20.

I compare, 2dly, the second price 24, which is greater than 22, and I again write the difference opposite 20 and by the

. If there were still a price less than 22, but greater
, we should place this difference opposite that price.
hich, I compare the least price 20 with 22, and
he difference 2 opposite the two first prices 30 and
h are both greater than 22. This done, I add the
es 2, 2, 8, 2; their sum is 14.
fore, to have a mixture the price of which shall be
must take two bushels at 30*s.*, two bushels at 24*s.*,
bushels at 20*s.*
ct, two bushels at 30*s.* give 60*s.*, two bushels at 24*s.*
., and ten bushels at 20*s.* give 200*s.* I add the num-
48, and 200, and I divide the sum 308*s.* by 14, which
mber of the bushels in the mixture. I find 22*s.* for
ent; but 14 bushels at 22*s.* make 308*s.*, that is to
+48*s.*+200*s.*; therefore 22*s.* is the price of a bushel
ixture.
se at present, that we would have a mixtue of 100
To know how many we must take of wheat, rye,
ey, we shall say:
nake a mixture of 14 bushels, we take 2 bushels of
ow many of wheat must we take for a mixture of
els, that is to say,

	bush.		*bush.*		*bush.*		
	14	:	2	::	100	:	$x=14+\frac{2}{7}$
I have	14	:	2	::	100	:	$y=14+\frac{2}{7}$
	14	:	10	::	100	:	$z=71+\frac{3}{7}$
					Sum		100 bushels.

ould prove this as above.

OF BARTER.

we make the exchange of one merchandise for
it is by the gain or interest of the moneys that we
e value of the merchandises, the gain or the loss.

Example I.

nerchants would make an exchange of merchandise;
has spices which are only worth 9*s.* per lb: ready
but in exchange he would have 10*s.* per lb. The

other merchant has wax which is worth 12*s.* per lb. ready money. He would know how he should sell it so as not to lose.

We shall say, if 9*s.* ready money be worth 10*s.* in exchange, how much is 12*s.* ready money worth in exchange? That is to say,

$$9s. : 10s. :: 12s. : x = 13 + \tfrac{1}{3} \text{ Ans.}$$

Example II.

Two merchants would make an exchange of merchandise; the one has serge, which is worth 56*cts.* per yard ready money, and in exchange, he would have 60*cts.* per yard, he would also have the third in ready money. The other merchant has wool which is worth 20*cts.* per lb. ready money. He would know how he must sell it so as not to lose?

I take the third of 60 which is 20, I take 20 from 56 and from 60; there will remain 36 of the first and 40 of the second; and we shall say, if 36 cts. ready money are worth 40 cts. in exchange, how much are 20 cts. ready money worth? that is to say,

$$\begin{array}{ccccc} cts. & cts. & & cts. & \\ 36 & : 40 & :: & 20 & : x = 22\tfrac{2}{9} \text{ Ans.} \end{array}$$

Example III.

Two merchants would exchange merchandise. The one has tin worth 8 cts. per lb. ready money, and in exchange he would have 10 cts. per lb.; the other merchant has brass which is worth 26 cts. ready money, and in exchange, he would have 30 cts. per lb. We would know which of the two gains?

We shall say, if 8 cts. ready money are worth 10 cts. in exchange, how much are 26 cts. ready money worth in exchange? that is to say,

$$\begin{array}{ccccc} cts. & cts. & & cts. & cts. \\ 8 & : 10 & :: & 26 & : x = 32 + \tfrac{1}{2}. \end{array}$$

Therefore the brass merchant loses $2 + \frac{1}{2}$ cts. per lb., and the other gains them.

But if the brass merchant would have the third in ready money, which of the two gains?

I subtract the third of 30 which is 10, from 26 and from 30, there remains 16 of the first and 20 of the second, and we shall say,

$$16 : 20 :: 26 : x = 32 + \tfrac{1}{2}.$$

Therefore there is no loss sustained by either.

OF DUODECIMALS.

This rule is used in measuring cord-wood, artificers' work, etc. The foot is the principal measure, and is divided as follows:

1 foot, *ft.*	is equal to	12 inches or primes, marked thus	$'$	$\frac{1}{12}$
1 inch	-	12 parts or seconds, -	$''$	$(\frac{1}{12})^2$
1 second	-	12 thirds - - -	$'''$	$(\frac{1}{12})^3$
1 third	-	12 fourths - - -	$''''$	$(\frac{1}{12})^4$

etc.

Here we see that the inch or prime is the $\frac{1}{12}$ of a foot; the second is $\frac{1}{12}$ of $\frac{1}{12}$, or the $\frac{1}{144}$ of a foot; the third is the $\frac{1}{12}$ of $\frac{1}{144}$ or the $\frac{1}{1728}$, etc. These parts, from their regular ratio of 12, have obtained the name of duodecimals, which name is derived from the latin word *duodecim*, signifying 12.

OF THE MULTIPLICATION OF DUODECIMALS.

From the nature of fractions, we shall easily perceive what parts will be produced in multiplying together any of the above denominations. For example, if we multiply by feet, as these are whole numbers having a unit for their denominator, it is evident that the denominator of the part multiplied will not be altered: thus 3 feet multiplied by 2 feet will give 6 feet, that is to say, $\frac{3}{1} \times \frac{2}{1} = \frac{6}{1} = 6$ feet; 3 inches multiplied by 2 feet, that is to say, $\frac{3}{12} \times \frac{2}{1} = \frac{6}{12}$, or 6 inches; again 3 seconds multiplied by 2 feet, or $\frac{3}{144} \times \frac{2}{1} = \frac{6}{144}$, or 6 seconds, etc.

If we multiply by inches, as the denominator of these is 12, the denominator of the part multiplied will become the same as that of the next lower denomination: thus 3 feet multiplied by 2 inches will give 6 inches, that is to say, $\frac{3}{1} \times \frac{2}{12} = \frac{6}{12}$, or 6 inches; 3 inches multiplied by 2 inches, that is to say, $\frac{3}{12} \times \frac{2}{12} = \frac{6}{144}$, or 6 seconds; 3 seconds multiplied by 2 inches, or $\frac{3}{144} \times \frac{2}{12} = \frac{6}{1728}$, or 6 thirds, etc. From what has been said,

the products of any other parts will be easily estimated by one who has learned fractions.

Also, as the second is the square of the prime, the third the cube, etc. considering the feet as having 0 for their index, the inches 1, the seconds 2, the thirds 3, etc. when we multiply these together, the parts produced will have for their index the sum of the indices of the parts multiplied; thus the index of the prime is 1, that of the second is 2, the sum of these is 3, which is the index of the parts produced when we multiply inches by seconds. It is the same for any other parts.

EXAMPLE.

Let it be required to multiply 7 feet 3 inches 4 seconds, by 2 feet 4 inches. I write the numbers thus:

ft.			
7	3′	4″	
2°	4′		
14	6′	8″	
2	5′	1″	4‴
16	11′	9″	4‴

and I first multiply all the parts of the multiplicand by the feet of the multiplier. Beginning with the lowest denomination, which is here seconds, I say $2^\circ \times 4'' = 8''$, which I place under seconds; then $2^\circ \times 3' = 6'$, which I write under inches. Lastly, $2^\circ \times 7^\circ = 14^\circ$ feet, which I write under feet.

Again, to multiply by 4′, I say $4' \times 4'' = 16'''$, and as these are equal to $1'' + 4'''$, I write 4‴ one place farther to the right and retain the 1″ to add it to the next product; then $4' \times$ $= 12''$, and adding the 1″ which I retain, I have 13″; these are equal to $1' + 1''$, wherefore I write the 1″ under 8″, and retain the 1′ to add it to the next product; lastly, $4' \times 7^\circ = 28$ and adding the 1′ I have 29′; these are equal to 2 feet 5′; therefore write 5′ under the primes or inches, and the two feet under feet. Adding the two products, I have 16 feet, 11 primes, 9 seconds, 4 thirds for the total product.

TO FIND THE SPACE CONTAINED IN ANY RIGHT ANGLED FIGURE OF FOUR SIDES.

Multiply the length taken in feet, inches, or any other measure by the breadth, taken in the same measure, and the product will be the number of square feet, square inches, or whatever measure you employed, contained in the figure.

The reason of this will be seen from the following figure.

The length of which, A B taken in feet, inches, or any other measure, we shall suppose to be 6, and the breadth A C taken in the same measure to be 3. Now supposing this measure to be a foot, it is evident that for every foot in breadth we shall have 6 square feet, there will then be 3 times 6 square feet, or 18 square feet, in the figure A B C D, and it is the same for any other measure.

A TABLE OF LOGARITHMS.

No. 1—100								Log. 1.000000—2.000000	
No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556302	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

No. 1000—1600 Log. 000000—204120

No.	0	1	2	3	4	5	6	7	8	9	Diff.
100	000000	000434	000868	001301	001734	002166	002598	003029	003460	003891	432
101	04321	04751	05180	05609	06038	06466	06894	07321	07748	08174	428
102	08600	09026	09451	09876	10300	10724	11147	11570	11993	12415	424
103	12837	13259	13680	14100	14520	14940	15360	15779	16197	16615	420
104	17033	17451	17868	18284	18700	19116	19532	19947	20361	20775	416
105	21189	21603	22016	22428	22841	23252	23664	24075	24486	24896	412
106	25306	25715	26124	26533	26942	27350	27757	28164	28571	28978	408
107	29384	29789	30195	30600	31004	31408	31812	32216	32619	33021	404
108	33424	33826	34227	34628	35029	35430	35830	36229	36629	37028	400
109	37426	37825	38223	38620	39017	39414	39811	40207	40602	40998	397
110	41393	41787	42182	42575	42969	43362	43755	44148	44540	44931	393
111	45323	45713	46105	46495	46885	47275	47664	48053	48442	48830	390
112	49218	49606	49993	50380	50766	51152	51538	51924	52309	52694	386
113	53078	53463	53846	54230	54613	54996	55378	55760	56142	56524	383
114	56905	57286	57666	58046	58426	58805	59185	59563	59942	60320	379
115	60698	61075	61452	61829	62206	62582	62958	63333	63709	64083	376
116	64458	64832	65206	65580	65953	66326	66699	67071	67443	67814	373
117	68186	68557	68928	69298	69668	70038	70407	70776	71145	71514	370
118	71882	72250	72617	72985	73352	73718	74085	74451	74816	75182	366
119	75547	75912	76276	76640	77004	77368	77731	78094	78457	78819	363
120	79181	79543	79904	80266	80626	80987	81347	81707	82066	82426	360
121	82785	83144	83503	83861	84219	84576	84934	85291	85647	86004	357
122	86360	86716	87071	87426	87781	88136	88490	88845	89198	89552	355
123	89905	90258	90611	90963	91315	91667	92018	92370	92721	93071	352
124	93422	93772	94122	94471	94820	95169	95518	95866	96215	96562	349
125	96910	97257	97604	97951	98297	98644	98990	99335	99681	100026	346
126	100370	100715	101059	101403	101747	102090	102434	102777	103119	03462	343
127	03804	04146	04487	04828	05169	05510	05851	06191	06531	06870	341
128	07210	07549	07888	08227	08565	08903	09241	09578	09916	10253	338
129	10590	10926	11262	11598	11934	12270	12605	12940	13275	13609	335
130	13943	14277	14611	14944	15278	15610	15943	16276	16608	16940	332
131	17271	17603	17934	18265	18595	18926	19256	19586	19915	20245	330
132	20574	20903	21231	21560	21888	22216	22543	22871	23198	23525	328
133	23852	24178	24504	24830	25156	25481	25806	26131	26456	26781	325
134	27105	27429	27752	28076	28399	28722	29045	29368	29690	30012	323
135	30334	30655	30977	31298	31619	31939	32260	32580	32900	33219	321
136	33539	33858	34177	34496	34814	35133	35451	35768	36086	36403	318
137	36721	37037	37354	37670	37987	38303	38618	38934	39249	39564	316
138	39879	40194	40508	40822	41136	41450	41763	42076	42389	42702	314
139	43015	43327	43639	43951	44263	44574	44885	45196	45507	45818	311
140	46128	46438	46748	47058	47367	47676	47985	48294	48603	48911	309
141	49219	49527	49835	50142	50449	50756	51063	51370	51676	51982	307
142	52288	52594	52900	53205	53510	53815	54119	54424	54728	55032	305
143	55336	55640	55943	56246	56549	56852	57154	57457	57759	58061	303
144	58362	58664	58965	59266	59567	59868	60168	60468	60769	61068	300
145	61368	61667	61967	62266	62564	62863	63161	63460	63757	64055	298
146	64353	64650	64947	65244	65541	65838	66134	66430	66726	67022	296
147	67317	67613	67908	68203	68497	68792	69086	69380	69674	69968	294
148	70262	70555	70848	71141	71434	71726	72019	72311	72603	72895	292
149	73186	73478	73769	74060	74351	74641	74932	75222	75512	75802	290
150	76091	76381	76670	76959	77248	77536	77825	78113	78401	78689	288
151	78977	79264	79552	79839	80126	80413	80699	80986	81272	81558	287
152	81844	82129	82415	82700	82985	83270	83554	83839	84123	84407	285
153	84691	84975	85259	85542	85825	86108	86391	86674	86956	87239	283
154	87521	87803	88084	88366	88647	88928	89209	89490	89771	90051	281
155	90332	90612	90892	91171	91451	91730	92010	92289	92567	92846	279
156	93125	93403	93681	93959	94237	94514	94792	95069	95346	95623	278
157	95900	96176	96452	96729	97005	97281	97556	97832	98107	98382	276
158	98657	98932	99206	99481	99755	200029	200303	200577	200850	201124	274
159	201397	201670	201943	202216	202488	02761	03033	03305	03577	03848	272
	0	1	2	3	4	5	6	7	8	9	

No. 1600—2200											Log. 204120—342423
No.	0	1	2	3	4	5	6	7	8	9	Diff.
160	204120	204391	204662	204933	205204	205475	205745	206016	206286	206556	271
161	06826	07095	07365	07634	07903	08172	08441	08710	08978	09247	269
162	09515	09783	10051	10318	10586	10853	11120	11388	11654	11921	267
163	12188	12454	12720	12986	13252	13518	13783	14049	14314	14579	266
164	14844	15109	15373	15638	15902	16166	16430	16694	16957	17221	264
165	17484	17747	18010	18273	18535	18798	19060	19322	19584	19846	262
166	20108	20370	20631	20892	21153	21414	21675	21936	22196	22456	261
167	22716	22976	23236	23496	23755	24015	24274	24533	24792	25051	259
168	25309	25568	25826	26084	26342	26600	26858	27115	27372	27630	258
169	27887	28144	28400	28657	28913	29170	29426	29682	29938	30193	256
170	30449	30704	30960	31215	31470	31724	31979	32233	32488	32742	255
171	32996	33250	33504	33757	34011	34264	34517	34770	35023	35276	253
172	35528	35781	36033	36285	36537	36789	37041	37292	37544	37795	252
173	38046	38297	38548	38799	39049	39299	39550	39800	40050	40300	250
174	40549	40799	41048	41297	41546	41795	42044	42293	42541	42790	249
175	43038	43286	43534	43782	44030	44277	44524	44772	45019	45266	247
176	45513	45759	46006	46252	46499	46745	46991	47236	47482	47728	246
177	47973	48219	48464	48709	48954	49198	49443	49687	49932	50176	245
178	50420	50664	50908	51151	51395	51638	51881	52125	52367	52610	243
179	52853	53096	53338	53580	53822	54064	54306	54548	54790	55031	242
180	55273	55514	55755	55996	56236	56477	56718	56958	57198	57439	241
181	57679	57918	58158	58398	58637	58877	59116	59355	59594	59833	239
182	60071	60310	60548	60787	61025	61263	61501	61738	61976	62214	238
183	62451	62688	62925	63162	63399	63636	63873	64109	64345	64582	237
184	64818	65054	65290	65525	65761	65996	66232	66467	66702	66937	235
185	67172	67406	67641	67875	68110	68344	68578	68812	69046	69279	234
186	69513	69746	69980	70213	70446	70679	70912	71144	71377	71609	233
187	71842	72074	72306	72538	72770	73001	73233	73464	73696	73927	232
188	74158	74389	74620	74850	75081	75311	75542	75772	76002	76232	230
189	76462	76691	76921	77151	77380	77609	77838	78067	78296	78525	229
190	78754	78982	79210	79439	79667	79895	80123	80351	80578	80806	228
191	81033	81261	81488	81715	81942	82169	82395	82622	82849	83075	227
192	83301	83527	83753	83979	84205	84431	84656	84882	85107	85332	226
193	85557	85782	86007	86232	86456	86681	86905	87130	87354	87578	225
194	87802	88025	88249	88473	88696	88920	89143	89366	89589	89812	223
195	90035	90257	90480	90702	90925	91147	91369	91591	91813	92034	222
196	92256	92478	92699	92920	93141	93363	93583	93804	94025	94246	221
197	94466	94687	94907	95127	95347	95567	95787	96007	96226	96446	220
198	96665	96884	97104	97323	97542	97761	97979	98198	98416	98635	219
199	98853	99071	99289	99507	99725	99943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	02331	02547	02764	02980	217
201	03196	03412	03628	03844	04059	04275	04491	04706	04921	05136	216
202	05351	05566	05781	05996	06211	06425	06639	06854	07068	07282	214
203	07496	07710	07924	08137	08351	08564	08778	08991	09204	09417	213
204	09630	09843	10056	10268	10481	10693	10906	11118	11330	11542	212
205	11754	11966	12177	12389	12600	12812	13023	13234	13445	13656	211
206	13867	14078	14289	14499	14710	14920	15130	15340	15550	15760	210
207	15970	16180	16390	16599	16809	17018	17227	17436	17645	17854	209
208	18063	18272	18481	18689	18898	19106	19314	19522	19730	19938	208
209	20146	20354	20562	20769	20977	21184	21391	21598	21805	22012	207
210	22219	22426	22633	22839	23046	23252	23458	23665	23871	24077	206
211	24282	24488	24694	24899	25105	25310	25516	25721	25926	26131	205
212	26336	26541	26745	26950	27155	27359	27563	27767	27972	28176	204
213	28380	28583	28787	28991	29194	29398	29601	29805	30008	30211	203
214	30414	30617	30819	31022	31225	31427	31630	31832	32034	32236	202
215	32438	32640	32842	33044	33246	33447	33649	33850	34051	34253	202
216	34454	34655	34856	35056	35257	35458	35658	35859	36059	36260	201
217	36460	36660	36860	37060	37260	37459	37659	37858	38058	38257	200
218	38456	38656	38855	39054	39253	39451	39650	39849	40047	40246	199
219	40444	40642	40841	41039	41237	41435	41632	41830	42028	42225	198
	0	1	2	3	4	5	6	7	8	9	

No. 2200—2800. Log. 342423—447158.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
220	342423	342620	342817	343014	343212	343409	343606	343802	343999	344196	197
221	44392	44589	44785	44981	45178	45374	45570	45766	45962	46157	196
222	46353	46549	46744	46939	47135	47330	47525	47720	47915	48110	195
223	48305	48500	48694	48889	49083	49278	49472	49666	49860	50054	194
224	50248	50442	50636	50829	51023	51216	51410	51603	51796	51989	193
225	52182	52375	52568	52761	52954	53147	53339	53532	53724	53916	193
226	54108	54301	54493	54685	54876	55068	55260	55452	55643	55834	192
227	56026	56217	56408	56599	56790	56981	57172	57363	57554	57744	191
228	57935	58125	58316	58506	58696	58886	59076	59266	59456	59646	190
229	59835	60025	60215	60404	60593	60783	60972	61161	61350	61539	189
230	61728	61917	62105	62294	62482	62671	62859	63048	63236	63424	188
231	63612	63800	63988	64176	64363	64551	64739	64926	65113	65301	188
232	65488	65675	65862	66049	66236	66423	66610	66796	66983	67169	187
233	67356	67542	67729	67915	68101	68287	68473	68659	68844	69030	186
234	69216	69401	69587	69772	69958	70143	70328	70513	70698	70883	185
235	71068	71253	71437	71622	71806	71991	72175	72360	72544	72728	184
236	72912	73096	73280	73464	73647	73831	74015	74198	74382	74565	184
237	74748	74932	75115	75298	75481	75664	75846	76029	76212	76394	183
238	76577	76759	76942	77124	77306	77488	77670	77852	78034	78216	182
239	78398	78580	78761	78943	79124	79306	79487	79668	79849	80030	181
240	80211	80392	80573	80754	80934	81115	81296	81476	81656	81837	181
241	82017	82197	82377	82557	82737	82917	83097	83277	83456	83636	180
242	83815	83995	84174	84353	84533	84712	84891	85070	85249	85428	179
243	85606	85785	85964	86142	86321	86499	86677	86856	87034	87212	178
244	87390	87568	87746	87923	88101	88279	88456	88634	88811	88989	178
245	89166	89343	89520	89697	89875	90051	90228	90405	90582	90759	177
246	90935	91112	91288	91464	91641	91817	91993	92169	92345	92521	176
247	92697	92873	93048	93224	93400	93575	93751	93926	94101	94277	176
248	94452	94627	94802	94977	95152	95326	95501	95676	95850	96025	175
249	96199	96374	96548	96722	96896	97071	97245	97418	97592	97766	174
250	97940	98114	98287	98461	98634	98808	98981	99154	99327	99501	173
251	99674	99847	400020	400192	400365	400538	400711	400883	401056	401228	173
252	401400	401573	01745	01917	02089	02261	02433	02605	02777	02949	172
253	03120	03292	03464	03635	03807	03978	04149	04320	04492	04663	171
254	04834	05005	05175	05346	05517	05688	05858	06029	06199	06370	171
255	06540	06710	06881	07051	07221	07391	07561	07731	07900	08070	170
256	08240	08410	08579	08749	08918	09087	09257	09426	09595	09764	169
257	09933	10102	10271	10440	10608	10777	10946	11114	11283	11451	169
258	11620	11788	11956	12124	12292	12460	12628	12796	12964	13132	168
259	13300	13467	13635	13802	13970	14137	14305	14472	14639	14806	167
260	14973	15140	15307	15474	15641	15808	15974	16141	16308	16474	167
261	16640	16807	16973	17139	17306	17472	17638	17804	17970	18135	166
262	18301	18467	18633	18798	18964	19129	19295	19460	19625	19791	165
263	19956	20121	20286	20451	20616	20781	20945	21110	21275	21439	165
264	21604	21768	21933	22097	22261	22426	22590	22754	22918	23082	164
265	23246	23410	23573	23737	23901	24064	24228	24392	24555	24718	164
266	24882	25045	25208	25371	25534	25697	25860	26023	26186	26349	163
267	26511	26674	26836	26999	27161	27324	27486	27648	27811	27973	162
268	28135	28297	28459	28621	28782	28944	29106	29268	29429	29591	162
269	29752	29914	30075	30236	30398	30559	30720	30881	31042	31203	161
270	31364	31525	31685	31846	32007	32167	32328	32488	32649	32809	160
271	32969	33129	33290	33450	33610	33770	33930	34090	34249	34409	160
272	34569	34728	34888	35048	35207	35366	35526	35685	35844	36003	159
273	36163	36322	36481	36640	36798	36957	37116	37275	37433	37592	159
274	37751	37909	38067	38226	38384	38542	38700	38859	39017	39175	158
275	39333	39491	39648	39806	39964	40122	40279	40437	40594	40752	158
276	40909	41066	41224	41381	41538	41695	41852	42009	42166	42323	157
277	42480	42636	42793	42950	43106	43263	43419	43576	43732	43888	157
278	44045	44201	44357	44513	44669	44825	44981	45137	45293	45448	156
279	45604	45760	45915	46071	46226	46382	46537	46692	46848	47003	155
	0	1	2	3	4	5	6	7	8	9	

No. 2800—3400 Log. 447158—531479

No.	0	1	2	3	4	5	6	7	8	9	Diff.
280	447158	447313	447468	447623	447778	447933	448088	448242	448397	448552	155
281	48706	48861	49015	49170	49324	49478	49633	49787	49941	50095	154
282	50249	50403	50557	50711	50865	51018	51172	51326	51479	51633	154
283	51786	51940	52093	52247	52400	52553	52706	52859	53012	53165	153
284	53318	53471	53624	53777	53930	54082	54235	54387	54540	54692	153
285	54845	54997	55149	55302	55454	55606	55758	55910	56062	56214	152
286	56366	56518	56670	56821	56973	57125	57276	57428	57579	57730	152
287	57882	58033	58184	58336	58487	58638	58789	58940	59091	59242	151
288	59392	59543	59694	59845	59995	60146	60296	60447	60597	60747	151
289	60898	61048	61198	61348	61498	61649	61799	61948	62098	62248	150
290	62398	62548	62697	62847	62997	63146	63296	63445	63594	63744	150
291	63893	64042	64191	64340	64489	64639	64787	64936	65085	65234	149
292	65383	65532	65680	65829	65977	66126	66274	66423	66571	66719	149
293	66868	67016	67164	67312	67460	67608	67756	67904	68052	68200	148
294	68347	68495	68643	68790	68938	69085	69233	69380	69527	69675	148
295	69822	69969	70116	70263	70410	70557	70704	70851	70998	71145	147
296	71292	71438	71585	71732	71878	72025	72171	72317	72464	72610	147
297	72756	72903	73049	73195	73341	73487	73633	73779	73925	74070	146
298	74216	74362	74508	74653	74799	74944	75090	75235	75381	75526	146
299	75671	75816	75962	76107	76252	76397	76542	76687	76832	76976	145
300	77121	77266	77411	77555	77700	77844	77989	78133	78278	78422	145
301	78566	78711	78855	78999	79143	79287	79431	79575	79719	79863	144
302	80007	80151	80294	80438	80582	80725	80869	81012	81156	81299	144
303	81443	81586	81729	81872	82016	82159	82302	82445	82588	82731	143
304	82874	83016	83159	83302	83445	83587	83730	83872	84015	84157	143
305	84300	84442	84584	84727	84869	85011	85153	85295	85437	85579	142
306	85721	85863	86005	86147	86289	86430	86572	86714	86855	86997	142
307	87138	87280	87421	87563	87704	87845	87986	88127	88269	88410	141
308	88551	88692	88833	88973	89114	89255	89396	89537	89677	89818	141
309	89958	90099	90239	90380	90520	90661	90801	90941	91081	91222	140
310	91362	91502	91642	91782	91922	92062	92201	92341	92481	92621	140
311	92760	92900	93040	93179	93319	93458	93597	93737	93876	94015	139
312	94155	94294	94433	94572	94711	94850	94989	95128	95267	95406	139
313	95544	95683	95822	95960	96099	96237	96376	96514	96653	96791	139
314	96930	97068	97206	97344	97482	97621	97759	97897	98035	98173	138
315	98311	98448	98586	98724	98862	98999	99137	99275	99412	99550	138
316	99687	99824	99962	500099	500236	500374	500511	500648	500785	500922	137
317	501059	501196	501333	01470	01607	01744	01880	02017	02154	02290	137
318	02427	02564	02700	02837	02973	03109	03246	03382	03518	03654	136
319	03791	03927	04063	04199	04335	04471	04607	04743	04878	05014	136
320	05150	05286	05421	05557	05692	05828	05963	06099	06234	06370	136
321	06505	06640	06775	06911	07046	07181	07316	07451	07586	07721	135
322	07856	07991	08125	08260	08395	08530	08664	08799	08933	09068	135
323	09202	09337	09471	09606	09740	09874	10008	10143	10277	10411	134
324	10545	10679	10813	10947	11081	11215	11348	11482	11616	11750	134
325	11883	12017	12150	12284	12417	12551	12684	12818	12951	13084	133
326	13218	13351	13484	13617	13750	13883	14016	14149	14282	14415	133
327	14548	14680	14813	14946	15079	15211	15344	15476	15609	15741	133
328	15874	16006	16139	16271	16403	16535	16668	16800	16932	17064	132
329	17196	17328	17460	17592	17724	17855	17987	18119	18251	18382	132
330	18514	18645	18777	18909	19040	19171	19303	19434	19565	19697	131
331	19828	19959	20090	20221	20352	20483	20614	20745	20876	21007	131
332	21138	21269	21400	21530	21661	21792	21922	22053	22183	22314	131
333	22444	22575	22705	22835	22966	23096	23226	23356	23486	23616	130
334	23746	23876	24006	24136	24266	24396	24526	24656	24785	24915	130
335	25045	25174	25304	25434	25563	25693	25822	25951	26081	26210	129
336	26339	26468	26598	26727	26856	26985	27114	27243	27372	27501	129
337	27630	27759	27888	28016	28145	28274	28402	28531	28660	28788	129
338	28917	29045	29174	29302	29430	29559	29687	29815	29943	30072	128
339	30200	30328	30456	30584	30712	30840	30968	31095	31223	31351	128
	0	1	2	3	4	5	6	7	8	9	

No. [illegible] Log. [illegible]—[illegible]

No.	0	1	2	3	4	5	6	7	8	9	Diff.
140	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13372	13500	13627	128
141	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13643	13772	13899	127
142	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14914	15041	15167	127
143	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	16306	16432	126
144	16557	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17707	17833	126
145	17959	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	19095	19221	126
146	19347	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	20473	20204	125
147	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21840	21454	125
148	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	23578	22701	125
149	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	23696	23820	23944	124
150	24067	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	24926	25050	25173	124
151	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	26172	26296	26419	124
152	26543	26666	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	27406	27529	27652	123
153	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28635	28758	28881	123
154	[illegible]	29128	[illegible]	[illegible]	[illegible]	[illegible]	29730	29861	29984	30106	123
155	30229	30351	[illegible]	[illegible]	[illegible]	[illegible]	30962	31084	31206	31328	122
156	31450	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	32181	32303	32425	32546	122
157	32668	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	33395	33516	33640	33762	121
158	33883	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	34731	34852	34973	35094	121
159	35094	35215	[illegible]	[illegible]	[illegible]	[illegible]	35940	36061	36182	36182	121
160	36302	36422	[illegible]	[illegible]	[illegible]	[illegible]	37026	37146	37267	37387	120
161	37507	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	38229	38348	38469	38589	120
162	38709	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	39425	39545	39667	39787	120
163	39907	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	40624	40743	40863	40982	119
164	41101	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	41817	41936	42055	42174	119
165	42293	42412	[illegible]	[illegible]	[illegible]	42769	42887	43006	43125	43244	119
166	43481	43600	[illegible]	[illegible]	[illegible]	[illegible]	44074	44192	44311	44429	119
167	44666	44784	[illegible]	[illegible]	[illegible]	45237	45378	45494	45612	45730	118
168	45848	45966	[illegible]	[illegible]	[illegible]	46417	46535	46653	46791	46909	118
169	47026	47144	[illegible]	[illegible]	[illegible]	47614	47732	47849	47967	48084	118
170	48202	48319	[illegible]	[illegible]	[illegible]	48788	48905	49023	49140	49257	117
171	49374	49491	[illegible]	[illegible]	[illegible]	49959	50076	50193	50309	50426	117
172	50543	50660	[illegible]	50993	51109	51126	51243	51359	51476	51592	117
173	51709	51825	[illegible]	[illegible]	52174	52291	52407	52523	52639	52755	116
174	52872	52988	[illegible]	[illegible]	53336	53453	53569	53684	53800	53915	116
175	54031	54147	[illegible]	54379	54494	54610	54726	54841	54957	55072	116
176	55188	55303	55419	55534	55650	55765	55880	55996	56111	56226	115
177	56341	56457	56572	56687	56802	56917	57032	57147	57262	57377	115
178	57492	57607	57721	57836	57951	58066	58181	58295	58410	58525	115
179	58639	58754	58868	58983	59097	59212	59326	59441	59555	59669	114
180	59784	59898	60012	60126	60240	60355	60469	60583	60697	60811	114
181	60925	61039	61153	61267	61381	61495	61608	61722	61836	61950	114
182	62063	62177	62291	62404	62518	62631	62745	62858	62972	63085	114
183	63199	63312	63426	63539	63652	63765	63879	63992	64105	64218	113
184	64331	64444	64557	64670	64783	64896	65009	65122	65235	65348	113
185	65461	65574	65686	65799	65912	66024	66137	66250	66362	66475	113
186	66587	66700	66812	66925	67037	67149	67262	67374	67486	67599	112
187	67711	67823	67935	68047	68160	68272	68384	68496	68608	68720	112
188	68832	68944	69056	69167	69279	69391	69503	69615	69726	69838	112
189	69950	70061	70173	70284	70396	70507	70619	70730	70842	70953	112
190	71065	71176	71287	71399	71510	71621	71732	71843	71955	72066	111
191	72177	72288	72399	72510	72621	72732	72843	72954	73064	73175	111
192	73286	73397	73508	73618	73729	73840	73950	74061	74171	74282	111
193	74393	74503	74613	74724	74834	74945	75055	75165	75276	75386	110
194	75496	75606	75717	75827	75937	76047	76157	76267	76377	76487	110
195	76597	76707	76817	76927	77037	77146	77256	77366	77476	77585	110
196	77695	77805	77914	78024	78134	78243	78353	78462	78572	78681	110
197	78790	78900	79009	79119	79228	79337	79446	79556	79665	79774	109
198	79883	79992	80101	80210	80319	80428	80537	80646	80755	80864	109
199	80973	81082	81190	81299	81408	81517	81625	81734	81843	81951	109
	0	1	2	3	4	5	6	7	8	9	

No. 4000—4600 Log. 602060—662758

No.	0	1	2	3	4	5	6	7	8	9	Diff.
400	602060	602169	602277	602386	602494	602603	602711	602819	602928	603036	108
401	03144	03253	03361	03469	03577	03686	03794	03902	04010	04118	108
402	04226	04334	04442	04550	04658	04766	04874	04982	05089	05197	108
403	05305	05413	05521	05628	05736	05844	05951	06059	06166	06274	108
404	06381	06489	06596	06704	06811	06919	07026	07133	07241	07348	107
405	07455	07562	07669	07777	07884	07991	08098	08205	08312	08419	107
406	08526	08633	08740	08847	08954	09061	09167	09274	09381	09488	107
407	09594	09701	09808	09914	10021	10128	10234	10341	10447	10554	107
408	10660	10767	10873	10979	11086	11192	11298	11405	11511	11617	106
409	11723	11829	11936	12042	12148	12254	12360	12466	12572	12678	106
410	12784	12890	12996	13101	13207	13313	13419	13525	13630	13736	106
411	13842	13947	14053	14159	14264	14370	14475	14581	14686	14792	106
412	14897	15003	15108	15213	15319	15424	15529	15634	15740	15845	105
413	15950	16055	16160	16265	16370	16475	16580	16685	16790	16895	105
414	17000	17105	17210	17315	17420	17524	17629	17734	17839	17943	105
415	18048	18153	18257	18362	18466	18571	18675	18780	18884	18989	105
416	19093	19198	19302	19406	19511	19615	19719	19823	19928	20032	104
417	20136	20240	20344	20448	20552	20656	20760	20864	20968	21072	104
418	21176	21280	21384	21488	21592	21695	21799	21903	22007	22110	104
419	22214	22318	22421	22525	22628	22732	22835	22939	23042	23146	104
420	23249	23353	23456	23559	23663	23766	23869	25972	24076	24179	103
421	24282	24385	24488	24591	24694	24798	24901	25004	25107	25209	103
422	25312	25415	25518	25621	25724	25827	25929	26032	26135	26238	103
423	26340	26443	26546	26648	26751	26854	26956	27058	27161	27263	103
424	27366	27468	27571	27673	27775	27878	27980	28082	28184	28287	102
425	28389	28491	28593	28695	28797	28900	29002	29104	29206	29308	102
426	29410	29511	29613	29715	29817	29919	30021	30123	30224	30326	102
427	30428	30530	30631	30733	30834	30936	31038	31139	31241	31342	102
428	31444	31545	31647	31748	31849	31951	32052	32153	32255	32356	101
429	32457	32558	32660	32761	32862	32963	33064	33165	33266	33367	101
430	33468	33569	33670	33771	33872	33973	34074	34175	34276	34376	101
431	34477	34578	34679	34779	34880	34981	35081	35182	35283	35383	101
432	35484	35584	35685	35785	35886	35986	36086	36187	36287	36388	100
433	36488	36588	36688	36789	36889	36989	37089	37189	37289	37390	100
434	37490	37590	37690	37790	37890	37990	38090	38190	38289	38389	100
435	38489	38589	38689	38789	38888	38988	39088	39188	39287	39387	100
436	39486	39586	39686	39785	39885	39984	40084	40183	40283	40382	99
437	40481	40581	40680	40779	40879	40978	41077	41176	41276	41375	99
438	41474	41573	41672	41771	41870	41970	42069	42168	42267	42366	99
439	42464	42563	42662	42761	42860	42959	43058	43156	43255	43354	99
440	43453	43551	43650	43749	43847	43946	44044	44143	44242	44340	98
441	44439	44537	44635	44734	44832	44931	45029	45127	45226	45324	98
442	45422	45520	45619	45717	45815	45913	46011	46109	46208	46306	98
443	46404	46502	46600	46698	46796	46894	46991	47089	47187	47285	98
444	47383	47481	47579	47676	47774	47872	47969	48067	48165	48262	98
445	48360	48458	48555	48653	48750	48848	48945	49043	49140	49237	97
446	49335	49432	49530	49627	49724	49821	49919	50016	50113	50210	97
447	50307	50405	50502	50599	50696	50793	50890	50987	51084	51181	97
448	51278	51375	51472	51569	51666	51762	51859	51956	52053	52150	97
449	52246	52343	52440	52536	52633	52730	52826	52923	53019	53116	97
450	53213	53309	53405	53502	53598	53695	53791	53888	53984	54080	96
451	54176	54273	54369	54465	54562	54658	54754	54850	54946	55042	96
452	55138	55234	55331	55427	55523	55619	55714	55810	55906	56002	96
453	56098	56194	56290	56386	56481	56577	56673	56769	56864	56960	96
454	57056	57151	57247	57343	57438	57534	57629	57725	57820	57916	96
455	58011	58107	58202	58298	58393	58488	58584	58679	58774	58870	95
456	58965	59060	59155	59250	59346	59441	59536	59631	59726	59821	95
457	59916	60011	60106	60201	60296	60391	60486	60581	60676	60771	95
458	60865	60960	61055	61150	61245	61339	61434	61529	61623	61718	95
459	61813	61907	62002	62096	62191	62285	62380	62474	62569	62663	95
	0	1	2	3	4	5	6	7	8	9	

No. 460—470. Log. 66276—71008

No.	0	1	2	3	4	5	6	7	8	9	Diff.
460	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	94
461	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	94
462	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	94
463	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	94
464	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	94
465	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	93
466	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	93
467	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	93
468	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	93
469	71173	71265	71356	71448	71540	71632	71724	71816	71913	72005	93
470	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	92
471	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	92
472	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	92
473	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	92
474	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	92
475	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91
476	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91
477	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91
478	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91
479	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91
480	81341	81432	81522	81613	81703	81794	81884	81974	81964	82055	90
481	82145	82235	82326	82416	82506	82596	82686	82777	82867	82957	90
482	83047	83137	83227	83317	83407	83497	83587	83677	83767	83857	90
483	83947	84037	84127	84217	84307	84396	84486	84576	84666	84756	90
484	84845	84935	85025	85114	85204	85294	85383	85473	85563	85652	90
485	85742	85831	85921	86010	86100	86189	86279	86368	86457	86547	89
486	86636	86725	86815	86904	86994	87083	87172	87261	87351	87440	89
487	87529	87618	87707	87796	87886	87975	88064	88153	88242	88331	89
488	88420	88509	88598	88687	88776	88865	88953	89042	89131	89220	89
489	89309	89398	89486	89575	89664	89753	89841	89930	90019	90107	89
490	90196	90285	90373	90462	90550	90639	90727	90816	90905	90993	89
491	91081	91170	91258	91347	91435	91524	91612	91700	91789	91877	88
492	91965	92053	92142	92230	92318	92406	92494	92583	92671	92759	88
493	92847	92935	93023	93111	93199	93287	93375	93463	93551	93639	88
494	93727	93815	93903	93991	94079	94166	94254	94342	94430	94517	88
495	94605	94693	94781	94868	94956	95044	95131	95219	95307	95394	88
496	95482	95569	95657	95744	95832	95919	96007	96094	96182	96269	87
497	96356	96444	96531	96618	96706	96793	96880	96968	97055	97142	87
498	97229	97317	97404	97491	97578	97665	97752	97839	97926	98013	87
499	98100	98188	98275	98362	98449	98535	98622	98709	98796	98883	87
500	99970	99057	99144	99231	99317	99404	99491	99578	99664	99751	87
501	99838	99924	700011	700098	700184	700271	700358	700444	700531	700617	87
502	700704	700790	00877	00963	01050	01136	01222	01309	01395	01482	86
503	01568	01654	01741	01827	01913	01999	02086	02172	02258	02344	86
504	02430	02517	02603	02689	02775	02861	02947	03033	03119	03205	86
505	03291	03377	03463	03549	03635	03721	03807	03893	03979	04065	86
506	04150	04236	04322	04408	04494	04579	04665	04751	04837	04922	86
507	05008	05094	05179	05265	05350	05436	05522	05607	05693	05778	86
508	05864	05949	06035	06120	06206	06291	06376	06462	06547	06632	85
509	06718	06803	06888	06974	07059	07144	07229	07315	07400	07485	85
510	07570	07655	07740	07826	07911	07996	08081	08166	08251	08336	85
511	08421	08506	08591	08676	08761	08846	08931	09015	09100	09185	85
512	09270	09355	09440	09524	09609	09694	09779	09863	09948	10033	85
513	10117	10202	10287	10371	10456	10540	10625	10710	10794	10879	85
514	10963	11048	11132	11216	11301	11385	11470	11554	11638	11723	84
515	11807	11892	11976	12060	12144	12229	12313	12397	12481	12566	84
516	12650	12734	12818	12902	12986	13070	13154	13238	13322	13406	84
517	13490	13574	13658	13742	13826	13910	13994	14078	14162	14246	84
518	14330	14414	14497	14581	14665	14749	14832	14916	15000	15084	84
519	15167	15251	15335	15418	15502	15586	15669	15753	15836	15920	84
	0	1	2	3	4	5	6	7	8	9	

No. 5200—5800 Log. 716003—763428

No.	0	1	2	3	4	5	6	7	8	9	Diff.
520	716003	716087	716170	716254	716337	716421	716504	716588	716671	716754	83
521	16838	16921	17004	17088	17171	17254	17338	17421	17504	17587	83
522	17671	17754	17837	17920	18003	18086	18169	18253	18336	18419	83
523	18502	18585	18668	18751	18834	18917	19000	19083	19165	19248	83
524	19331	19414	19497	19580	19663	19745	19828	19911	19994	20077	83
525	20159	20242	20325	20407	20490	20573	20655	20738	20821	20903	83
526	20986	21068	21151	21233	21316	21398	21481	21563	21646	21728	82
527	21811	21893	21975	22058	22140	22222	22305	22387	22469	22552	82
528	22634	22716	22798	22881	22963	23045	23127	23209	23291	23374	82
529	23456	23538	23620	23702	23784	23866	23948	24030	24112	24194	82
530	24276	24358	24440	24522	24603	24685	24767	24849	24931	25013	82
531	25095	25176	25258	25340	25422	25503	25585	25667	25748	25830	82
532	25912	25993	26075	26156	26238	26320	26401	26483	26564	26646	82
533	26727	26809	26890	26972	27053	27134	27216	27297	27379	27460	81
534	27541	27623	27704	27785	27866	27948	28029	28110	28191	28273	81
535	28354	28435	28516	28597	28678	28759	28841	28922	29003	29084	81
536	29165	29246	29327	29408	29489	29570	29651	29732	29813	29893	81
537	29974	30055	30136	30217	30298	30378	30459	30540	30621	30702	81
538	30782	30863	30944	31024	31105	31186	31266	31347	31428	31508	81
539	31589	31669	31750	31830	31911	31991	32072	32152	32233	32313	81
540	32394	32474	32555	32635	32715	32796	32876	32956	33037	33117	80
541	33197	33278	33358	33438	33518	33598	33679	33759	33839	33919	80
542	33999	34079	34159	34240	34320	34400	34480	34560	34640	34720	80
543	34800	34880	34960	35040	35120	35200	35279	35359	35439	35519	80
544	35599	35679	35759	35838	35918	35998	36078	36157	36237	36317	80
545	36396	36476	36556	36635	36715	36795	36874	36954	37034	37113	80
546	37193	37272	37352	37431	37511	37590	37670	37749	37829	37908	79
547	37987	38067	38146	38225	38305	38384	38463	38543	38622	38701	79
548	38781	38860	38939	39018	39097	39177	39256	39335	39414	39493	79
549	39572	39651	39730	39810	39889	39968	40047	40126	40205	40284	79
550	40363	40442	40521	40599	40678	40757	40836	40915	40994	41073	79
551	41152	41230	41309	41388	41467	41546	41624	41703	41782	41860	79
552	41939	42018	42096	42175	42254	42332	42411	42489	42568	42647	79
553	42725	42804	42882	42961	43039	43118	43196	43275	43353	43431	78
554	43510	43588	43667	43745	43823	43902	43980	44058	44136	44215	78
555	44293	44371	44449	44528	44606	44684	44762	44840	44919	44997	78
556	45075	45153	45231	45309	45387	45465	45543	45621	45699	45777	78
557	45855	45933	46011	46089	46167	46245	46323	46401	46479	46556	78
558	46634	46712	46790	46868	46945	47023	47101	47179	47256	47334	78
559	47412	47489	47567	47645	47722	47800	47878	47955	48033	48110	78
560	48188	48266	48343	48421	48498	48576	48653	48731	48808	48885	77
561	48963	49040	49118	49195	49272	49350	49427	49504	49582	49659	77
562	49736	49814	49891	49968	50045	50123	50200	50277	50354	50431	77
563	50508	50586	50663	50740	50817	50894	50971	51048	51125	51202	77
564	51279	51356	51433	51510	51587	51664	51741	51818	51895	51972	77
565	52048	52125	52202	52279	52356	52433	52509	52586	52663	52740	77
566	52816	52893	52970	53047	53123	53200	53277	53353	53430	53506	77
567	53583	53660	53736	53813	53889	53966	54042	54119	54195	54272	77
568	54348	54425	54501	54578	54654	54730	54807	54883	54960	55036	76
569	55112	55189	55265	55341	55417	55494	55570	55646	55722	55799	76
570	55875	55951	56027	56103	56180	56256	56332	56408	56484	56560	76
571	56636	56712	56788	56864	56940	57016	57092	57168	57244	57320	76
572	57396	57472	57548	57624	57700	57775	57851	57927	58003	58079	76
573	58155	58230	58306	58382	58458	58533	58609	58685	58761	58836	76
574	58912	58988	59063	59139	59214	59290	59366	59441	59517	59592	76
575	59668	59743	59819	59894	59970	60045	60121	60196	60272	60347	75
576	60422	60498	60573	60649	60724	60799	60875	60950	61025	61101	75
577	61176	61251	61326	61402	61477	61552	61627	61702	61778	61853	75
578	61928	62003	62078	62153	62228	62303	62378	62453	62529	62604	75
579	62679	62754	62829	62904	62978	63053	63128	63203	63278	63353	75
	0	1	2	3	4	5	6	7	8	9	

	0	1	2	3	4	5	6	7	8	9	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

No. 6400—7000 Log. 806180—845098

No.	0	1	2	3	4	5	6	7	8	9	Diff.
640	806180	806248	806316	806384	806451	806519	806587	806655	806723	806790	68
641	06858	06926	06994	07061	07129	07197	07264	07332	07400	07467	68
642	07535	07603	07670	07738	07806	07873	07941	08008	08076	08143	68
643	08211	08279	08346	08414	08481	08549	08616	08684	08751	08818	68
644	08886	08953	09021	09088	09156	09223	09290	09358	09425	09492	67
645	09560	09627	09694	09762	09829	09896	09964	10031	10098	10165	67
646	10233	10300	10367	10434	10501	10569	10636	10703	10770	10837	67
647	10904	10971	11038	11106	11173	11240	11307	11374	11441	11508	67
648	11575	11642	11709	11776	11843	11910	11977	12044	12111	12178	67
649	12245	12312	12378	12445	12512	12579	12646	12713	12780	12847	67
650	12913	12980	13047	13114	13181	13247	13314	13381	13448	13514	67
651	13581	13648	13714	13781	13848	13914	13981	14048	14114	14181	67
652	14248	14314	14381	14447	14514	14581	14647	14714	14780	14847	67
653	14913	14980	15046	15113	15179	15246	15312	15378	15445	15511	66
654	15578	15644	15711	15777	15843	15910	15976	16042	16109	16175	66
655	16241	16308	16374	16440	16506	16573	16639	16705	16771	16838	66
656	16904	16970	17036	17102	17169	17235	17301	17367	17433	17499	66
657	17565	17631	17698	17764	17830	17896	17962	18028	18094	18160	66
658	18226	18292	18358	18424	18490	18556	18622	18688	18754	18819	66
659	18885	18951	19017	19083	19149	19215	19281	19346	19412	19478	66
660	19544	19610	19675	19741	19807	19873	19939	20004	20070	20136	66
661	20201	20267	20333	20399	20464	20530	20595	20661	20727	20792	66
662	20858	20924	20989	21055	21120	21186	21251	21317	21382	21448	66
663	21514	21579	21644	21710	21775	21841	21906	21972	22037	22103	65
664	22168	22233	22299	22364	22430	22495	22560	22626	22691	22756	65
665	22822	22887	22952	23018	23083	23148	23213	23279	23344	23409	65
666	23474	23539	23605	23670	23735	23800	23865	23930	23996	24061	65
667	24126	24191	24256	24321	24386	24451	24516	24581	24646	24711	65
668	24776	24841	24906	24971	25036	25101	25166	25231	25296	25361	65
669	25426	25491	25556	25621	25686	25751	25815	25880	25945	26010	65
670	26075	26140	26204	26269	26334	26399	26464	26528	26593	26658	65
671	26723	26787	26852	26917	26981	27046	27111	27175	27240	27305	65
672	27369	27434	27498	27563	27628	27692	27757	27821	27886	27951	65
673	28015	28080	28144	28209	28273	28338	28402	28467	28531	28595	64
674	28660	28724	28789	28853	28918	28982	29046	29111	29175	29239	64
675	29304	29368	29432	29497	29561	29625	29690	29754	29818	29882	64
676	29947	30011	30075	30139	30204	30268	30332	30396	30460	30525	64
677	30589	30653	30717	30781	30845	30909	30973	31037	31102	31166	64
678	31230	31294	31358	31422	31486	31550	31614	31678	31742	31806	64
679	31870	31934	31998	32062	32126	32189	32253	32317	32381	32445	64
680	32509	32573	32637	32700	32764	32828	32892	32956	33020	33083	64
681	33147	33211	33275	33338	33402	33466	33530	33593	33657	33721	64
682	33784	33848	33912	33975	34039	34103	34166	34230	34293	34357	64
683	34421	34484	34548	34611	34675	34739	34802	34866	34929	34993	64
684	35056	35120	35183	35247	35310	35373	35437	35500	35564	35627	63
685	35691	35754	35817	35881	35944	36007	36071	36134	36197	36261	63
686	36324	36387	36451	36514	36577	36641	36704	36767	36830	36894	63
687	36957	37020	37083	37146	37210	37273	37336	37399	37462	37525	63
688	37588	37652	37715	37778	37841	37904	37967	38030	38093	38156	63
689	38219	38282	38345	38408	38471	38534	38597	38660	38723	38786	63
690	38849	38912	38975	39038	39101	39164	39227	39289	39352	39415	63
691	39478	39541	39604	39667	39729	39792	39855	39918	39981	40043	63
692	40106	40169	40232	40294	40357	40420	40482	40545	40608	40671	63
693	40733	40796	40859	40921	40984	41046	41109	41172	41234	41297	63
694	41359	41422	41485	41547	41610	41672	41735	41797	41860	41922	63
695	41985	42047	42110	42172	42235	42297	42360	42422	42484	42547	62
696	42609	42672	42734	42796	42859	42921	42983	43046	43108	43170	62
697	43233	43295	43357	43420	43482	43544	43606	43669	43731	43793	62
698	43855	43918	43980	44042	44104	44166	44229	44291	44353	44415	62
699	44477	44539	44601	44664	44726	44788	44850	44912	44974	45036	62
	0	1	2	3	4	5	6	7	8	9	

No. 7000—7600										Log. 845098—88081	
No.	0	1	2	3	4	5	6	7	8	9	Diff.
700	845098	845160	45222	845284	845346	845408	845470	845532	845594	845656	62
701	45718	45780	45842	45904	45966	46028	46090	46151	46213	46275	62
702	46337	46399	46461	46523	46584	46646	46708	46770	46832	46894	62
703	46955	47017	47079	47141	47202	47264	47326	47388	47449	47511	62
704	47573	47634	47696	47758	47819	47881	47943	48004	48066	48127	62
705	48189	48251	48312	48374	48435	48497	48559	48620	48682	48743	62
706	48805	48866	48928	48989	49051	49112	49174	49235	49296	49358	61
707	49419	49481	49542	49604	49665	49726	49788	49849	49911	49972	61
708	50033	50095	50156	50217	50279	50340	50401	50462	50524	50585	61
709	50646	50707	50769	50830	50891	50952	51014	51075	51136	51197	61
710	51258	51320	51381	51442	51503	51564	51625	51686	51747	51808	61
711	51870	51931	51992	52053	52114	52175	52236	52297	52358	52419	61
712	52480	52541	52602	52663	52724	52785	52846	52907	52968	53029	61
713	53090	53150	53211	53272	53333	53394	53455	53516	53576	53637	61
714	53698	53759	53820	53881	53941	54002	54063	54124	54185	54245	61
715	54306	54367	54427	54488	54549	54610	54670	54731	54792	54852	61
716	54913	54974	55034	55095	55156	55216	55277	55337	55398	55459	61
717	55519	55580	55640	55701	55761	55822	55882	55943	56003	56064	61
718	56124	56185	56245	56306	56366	56427	56487	56548	56608	56668	60
719	56729	56789	56850	56910	56970	57031	57091	57151	57212	57272	60
720	57332	57393	57453	57513	57574	57634	57694	57754	57815	57875	60
721	57935	57995	58056	58116	58176	58236	58297	58357	58417	58477	60
722	58537	58597	58657	58718	58778	58838	58898	58958	59018	59078	60
723	59138	59198	59258	59318	59378	59438	59499	59559	59619	59679	60
724	59739	59799	59858	59918	59978	60038	60098	60158	60218	60278	60
725	60338	60398	60458	60518	60578	60637	60697	60757	60817	60877	60
726	60937	60996	61056	61116	61176	61236	61295	61355	61415	61475	60
727	61534	61594	61654	61714	61773	61833	61893	61952	62012	62072	60
728	62131	62191	62251	62310	62370	62430	62489	62549	62608	62668	60
729	62728	62787	62847	62906	62966	63025	63085	63144	63204	63263	60
730	63323	63382	63442	63501	63561	63620	63680	63739	63798	63858	59
731	63917	63977	64036	64096	64155	64214	64274	64333	64392	64452	59
732	64511	64570	64630	64689	64748	64808	64867	64926	64985	65045	59
733	65104	65163	65222	65282	65341	65400	65459	65518	65578	65637	59
734	65696	65755	65814	65874	65933	65992	66051	66110	66169	66228	59
735	66287	66346	66405	66465	66524	66583	66642	66701	66760	66819	59
736	66878	66937	66996	67055	67114	67173	67232	67291	67350	67409	59
737	67467	67526	67585	67644	67703	67762	67821	67880	67939	67998	59
738	68056	68115	68174	68233	68292	68350	68409	68468	68527	68586	59
739	68644	68703	68762	68821	68879	68938	68997	69056	69114	69173	59
740	69232	69290	69349	69408	69466	69525	69584	69642	69701	69760	59
741	69818	69877	69935	69994	70053	70111	70170	70228	70287	70345	59
742	70404	70462	70521	70579	70638	70696	70755	70813	70872	70930	58
743	70989	71047	71106	71164	71223	71281	71339	71398	71456	71515	58
744	71573	71631	71690	71748	71806	71865	71923	71981	72040	72098	58
745	72156	72215	72273	72331	72389	72448	72506	72564	72622	72681	58
746	72739	72797	72855	72913	72972	73030	73088	73146	73204	73262	58
747	73321	73379	73437	73495	73553	73611	73669	73727	73785	73843	58
748	73902	73960	74018	74076	74134	74192	74250	74308	74366	74424	58
749	74482	74540	74598	74656	74714	74772	74830	74887	74945	75003	58
750	75061	75119	75177	75235	75293	75351	75409	75466	75524	75582	58
751	75640	75698	75756	75813	75871	75929	75987	76045	76102	76160	58
752	76218	76276	76333	76391	76449	76507	76564	76622	76680	76737	58
753	76795	76853	76910	76968	77026	77083	77141	77198	77256	77314	58
754	77371	77429	77486	77544	77602	77659	77717	77774	77832	77889	58
755	77947	78004	78062	78119	78177	78234	78292	78349	78407	78464	57
756	78522	78579	78637	78694	78751	78809	78866	78924	78981	79038	57
757	79096	79153	79211	79268	79325	79383	79440	79497	79555	79612	57
758	79669	79726	79784	79841	79898	79956	80013	80070	80127	80185	57
759	80242	80299	80356	80413	80471	80528	80585	80642	80699	80756	57
	0	1	2	3	4	5	6	7	8	9	

No. 7600—8200 Log. 880814—913814

No.	0	1	2	3	4	5	6	7	8	9	Diff.
760	880814	880871	880928	880985	881042	881099	881156	881213	881270	881328	57
761	81385	81442	81499	81556	81613	81670	81727	81784	81841	81898	57
762	81955	82012	82069	82126	82183	82240	82297	82354	82411	82468	57
763	82524	82581	82638	82695	82752	82809	82866	82923	82980	83037	57
764	83093	83150	83207	83264	83321	83377	83434	83491	83548	83605	57
765	83661	83718	83775	83832	83888	83945	84002	84059	84115	84172	57
766	84229	84285	84342	84399	84455	84512	84569	84625	84682	84739	57
767	84795	84852	84909	84965	85022	85078	85135	85192	85248	85305	57
768	85361	85418	85474	85531	85587	85644	85700	85757	85813	85870	57
769	85926	85983	86039	86096	86152	86209	86265	86321	86378	86434	56
770	86491	86547	86604	86660	86716	86773	86829	86885	86942	86998	56
771	87054	87111	87167	87223	87280	87336	87392	87449	87505	87561	56
772	87617	87674	87730	87786	87842	87898	87955	88011	88067	88123	56
773	88179	88236	88292	88348	88404	88460	88516	88573	88629	88685	56
774	88741	88797	88853	88909	88965	89021	89077	89134	89190	89246	56
775	89302	89358	89414	89470	89526	89582	89638	89694	89750	89806	56
776	89862	89918	89974	90030	90086	90141	90197	90253	90309	90365	56
777	90421	90477	90533	90589	90644	90700	90756	90812	90868	90924	56
778	90980	91035	91091	91147	91203	91259	91314	91370	91426	91482	56
779	91537	91593	91649	91705	91760	91816	91872	91928	91983	92039	56
780	92095	92150	92206	92262	92317	92373	92429	92484	92540	92595	56
781	92651	92707	92762	92818	92873	92929	92985	93040	93096	93151	56
782	93207	93262	93318	93373	93429	93484	93540	93595	93651	93706	56
783	93762	93817	93873	93928	93984	94039	94094	94150	94205	94261	55
784	94316	94371	94427	94482	94538	94593	94648	94704	94759	94814	55
785	94870	94925	94980	95036	95091	95146	95201	95257	95312	95367	55
786	95423	95478	95533	95588	95643	95699	95754	95809	95864	95920	55
787	95975	96030	96085	96140	96195	96251	96306	96361	96416	96471	55
788	96526	96581	96636	96692	96747	96802	96857	96912	96967	97022	55
789	97077	97132	97187	97242	97297	97352	97407	97462	97517	97572	55
790	97627	97682	97737	97792	97847	97902	97957	98012	98067	98122	55
791	98176	98231	98286	98341	98396	98451	98506	98561	98615	98670	55
792	98725	98780	98835	98890	98944	98999	99054	99109	99164	99218	55
793	99273	99328	99383	99437	99492	99547	99602	99656	99711	99766	55
794	99820	99875	99930	99985	900039	900094	900149	900203	900258	900312	55
795	900367	900422	900476	900531	00586	00640	00695	00749	00804	00858	55
796	00913	00968	01022	01077	01131	01186	01240	01295	01349	01404	55
797	01458	01513	01567	01622	01676	01731	01785	01840	01894	01948	54
798	02003	02057	02112	02166	02221	02275	02329	02384	02438	02491	54
799	02547	02601	02655	02710	02764	02818	02873	02927	02981	03036	54
800	03090	03144	03198	03253	03307	03361	03416	03470	03524	03578	54
801	03632	03687	03741	03795	03849	03903	03958	04012	04066	04120	54
802	04174	04228	04283	04337	04391	04445	04499	04553	04607	04661	54
803	04715	04770	04824	04878	04932	04986	05040	05094	05148	05202	54
804	05256	05310	05364	05418	05472	05526	05580	05634	05688	05742	54
805	05796	05850	05904	05958	06012	06065	06119	06173	06227	06281	54
806	06335	06389	06443	06497	06550	06604	06658	06712	06766	06820	54
807	06873	06927	06981	07035	07089	07142	07196	07250	07304	07358	54
808	07411	07465	07519	07573	07626	07680	07734	07787	07841	07895	54
809	07948	08002	08056	08109	08163	08217	08270	08324	08378	08431	54
810	08485	08539	08592	08646	08699	08753	08807	08860	08914	08967	54
811	09021	09074	09128	09181	09235	09288	09342	09395	09449	09502	54
812	09556	09609	09663	09716	09770	09823	09877	09930	09984	10037	53
813	10090	10144	10197	10251	10304	10358	10411	10464	10518	10571	53
814	10624	10678	10731	10784	10838	10891	10944	10998	11051	11104	53
815	11158	11211	11264	11317	11371	11424	11477	11530	11584	11637	53
816	11690	11743	11797	11850	11903	11956	12009	12063	12116	12169	53
817	12222	12275	12328	12381	12435	12488	12541	12594	12647	12700	53
818	12753	12806	12859	12913	12966	13019	13072	13125	13178	13231	53
819	13284	13337	13390	13443	13496	13549	13602	13655	13708	13761	53
	0	1	2	3	4	5	6	7	8	9	

No. 820—880 Log. 913914—944[illegible]

No.	0	1	2	3	4	5	6	7	8	9	Diff.
820	913914	913967	913920	913973	914026	914079	914131	914184	914237	914289	5
821	14343	14396	14449	14502	14555	14608	14660	14713	14766	14819	5
822	14872	14925	14977	15030	15083	15136	15189	15241	15294	15347	5
823	15400	15453	15505	15558	15611	15664	15716	15769	15822	15874	5
824	15927	15980	16033	16085	16138	16191	16243	16296	16349	16401	5
825	16454	16507	16559	16612	16664	16717	16770	16822	16875	16927	5
826	16980	17033	17085	17138	17190	17243	17295	17348	17400	17453	5
827	17505	17558	17610	17663	17715	17768	17820	17873	17925	17978	5
828	18030	18083	18135	18188	18240	18292	18345	18397	18450	18502	5
829	18555	18607	18659	18712	18764	18816	18869	18921	18973	19026	5
830	19078	19130	19183	19235	19287	19340	19392	19444	19496	19549	5
831	19601	19653	19705	19758	19810	19862	19914	19967	20019	20071	5
832	20123	20175	20228	20280	20332	20384	20436	20489	20541	20593	5
833	20645	20697	20749	20801	20853	20906	20958	21010	21062	21114	5
834	21166	21218	21270	21322	21374	21426	21478	21530	21582	21634	5
835	21686	21738	21790	21842	21894	21946	21998	22050	22102	22154	5
836	22206	22258	22310	22362	22414	22466	22519	22570	22622	22674	5
837	22725	22777	22829	22881	22933	22985	23037	23088	23140	23192	5
838	23244	23296	23348	23399	23451	23503	23555	23607	23658	23710	5
839	23762	23814	23865	23917	23969	24021	24072	24124	24176	24227	5
840	24279	24331	24383	24434	24486	24538	24589	24641	24693	24744	5
841	24796	24848	24899	24951	25002	25054	25106	25157	25209	25260	5
842	25312	25364	25415	25467	25518	25570	25621	25673	25724	25776	5
843	25828	25879	25931	25982	26034	26085	26137	26188	26239	26291	5
844	26342	26394	26445	26497	26548	26600	26651	26702	26754	26805	5
845	26857	26908	26959	27011	27062	27114	27165	27216	27268	27319	5
846	27370	27422	27473	27524	27576	27627	27678	27730	27781	27832	5
847	27883	27935	27986	28037	28088	28140	28191	28242	28293	28345	5
848	28396	28447	28498	28549	28601	28652	28703	28754	28805	28856	5
849	28908	28959	29010	29061	29112	29163	29214	29266	29317	29368	5
850	29419	29470	29521	29572	29623	29674	29725	29776	29827	29878	5
851	29930	29981	30032	30083	30134	30185	30236	30287	30338	30389	5
852	30440	30491	30541	30592	30643	30694	30745	30796	30847	30898	5
853	30949	31000	31051	31102	31153	31203	31254	31305	31356	31407	5
854	31458	31509	31560	31610	31661	31712	31763	31814	31864	31915	5
855	31966	32017	32068	32118	32169	32220	32271	32321	32372	32423	5
856	32474	32524	32575	32626	32677	32727	32778	32829	32879	32930	5
857	32981	33031	33082	33133	33183	33234	33285	33335	33386	33437	5
858	33487	33538	33588	33639	33690	33740	33791	33841	33892	33943	5
859	33993	34044	34094	34145	34195	34246	34296	34347	34397	34448	5
860	34498	34549	34599	34650	34700	34751	34801	34852	34902	34953	5
861	35003	35054	35104	35154	35205	35255	35306	35356	35406	35457	5
862	35507	35558	35608	[illegible]	35709	35759	35809	35860	35910	35960	5
863	36011	36061	36111	36162	[illegible]	36262	36313	36363	36413	36463	5
864	36514	36564	36614	36664	36715	36765	36815	36865	36916	36966	5
865	37016	37066	37116	37167	37217	37267	37317	37367	37418	37468	5
866	37518	37568	37618	37668	37718	37769	37819	37869	37919	37969	5
867	38019	38069	38119	38169	38219	38269	38319	38370	38420	38470	5
868	38520	38570	38620	38670	38720	38770	38820	38870	38920	38970	5
869	39020	39070	39120	39170	39220	39270	39319	39369	39419	39469	5
870	39519	39569	39619	39669	39719	39769	39819	39868	39918	39968	5
871	40018	40068	40118	40168	40218	40267	40317	40367	40417	40467	5
872	40516	40566	40616	40666	40716	40765	40815	40865	40915	40964	5
873	41014	41064	41113	41163	41213	41263	41313	41362	41412	41462	5
874	41511	41561	41611	41660	41710	41760	41809	41859	41909	41958	5
875	42008	42058	42107	42157	42206	42256	42306	42355	42405	42454	5
876	42504	42554	42603	42653	42702	42752	42801	42851	42900	42950	5
877	43000	43049	43099	43148	43198	43247	43297	43346	43396	43445	4
878	43494	43544	43593	43643	43692	43742	43791	43841	43890	43939	4
879	43989	44038	44088	44137	44186	44236	44285	44335	44384	44433	4
	0	1	2	3	4	5	6	7	8	9	

No. 8800—9400 Log. 944483—973128

No.	0	1	2	3	4	5	6	7	8	9	Diff.
880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927	49
881	44976	45025	45074	45124	45173	45222	45272	45321	45370	45419	49
882	45469	45518	45567	45616	45665	45715	45764	45813	45862	45911	49
883	45961	46010	46059	46108	46157	46207	46256	46305	46354	46403	49
884	46452	46501	46550	46600	46649	46698	46747	46796	46845	46894	49
885	46943	46992	47041	47090	47139	47189	47238	47287	47336	47385	49
886	47434	47483	47532	47581	47630	47679	47728	47777	47826	47875	49
887	47924	47973	48021	48070	48119	48168	48217	48266	48315	48364	49
888	48413	48462	48511	48560	48608	48657	48706	48755	48804	48852	49
889	48902	48951	48999	49048	49097	49146	49195	49244	49292	49341	49
890	49390	49439	49488	49536	49585	49634	49683	49731	49780	49829	49
891	49878	49926	49975	50024	50073	50121	50170	50219	50267	50316	49
892	50365	50413	50462	50511	50560	50608	50657	50705	50754	50803	49
893	50851	50900	50949	50997	51046	51095	51143	51192	51240	51289	49
894	51337	51386	51435	51483	51532	51580	51629	51677	51726	51774	49
895	51823	51872	51920	51969	52017	52066	52114	52163	52211	52259	48
896	52308	52356	52405	52453	52502	52550	52599	52647	52696	52744	48
897	52792	52841	52889	52938	52986	53034	53086	53131	53180	53228	48
898	53276	53325	53373	53421	53470	53518	53566	53615	53663	53711	48
899	53760	53808	53856	53905	53953	54001	54049	54098	54146	54194	48
900	54242	54291	54339	54387	54435	54484	54532	54580	54628	54677	48
901	54725	54773	54821	54869	54918	54966	55014	55062	55110	55158	48
902	55206	55255	55303	55351	55399	55447	55495	55543	55592	55640	48
903	55688	55736	55784	55832	55880	55928	55976	56024	56072	56120	48
904	56168	56216	56264	56312	56361	56409	56457	56505	56553	56601	48
905	56649	56697	56745	56792	56840	56888	56936	56984	57032	57080	48
906	57128	57176	57224	57272	57320	57368	57416	57464	57511	57559	48
907	57607	57655	57703	57751	57799	57847	57894	57942	57990	58038	48
908	58086	58134	58181	58229	58277	58325	58373	58420	58468	58516	48
909	58564	58612	58659	58707	58755	58803	58850	58898	58946	58994	48
910	59041	59089	59137	59184	59232	59280	59328	59375	59423	59471	48
911	59518	59566	59614	59661	59709	59757	59804	59852	59900	59947	48
912	59995	60042	60090	60138	60185	60233	60281	60328	60376	60423	48
913	60471	60518	60566	60613	60661	60709	60756	60804	60851	60899	48
914	60946	60994	61041	61089	61136	61184	61231	61279	61326	61374	48
915	61421	61469	61516	61563	61611	61658	61706	61753	61801	61848	47
916	61895	61943	61990	62038	62085	62132	62180	62227	62275	62322	47
917	62369	62417	62464	62511	62559	62606	62653	62701	62748	62795	47
918	62843	62890	62937	62985	63032	63079	63126	63174	63221	63268	47
919	63315	63363	63410	63457	63504	63552	63599	63646	63693	63741	47
920	63788	63835	63882	63929	63977	64024	64071	64118	64165	64212	47
921	64260	64307	64354	64401	64448	64495	64542	64590	64637	64684	47
922	64731	64778	64825	64872	64919	64966	65013	65060	65108	65155	47
923	65202	65249	65296	65343	65390	65437	65484	65531	65578	65625	47
924	65672	65719	65766	65813	65860	65907	65954	66001	66048	66095	47
925	66142	66189	66236	66283	66329	66376	66423	66470	66517	66564	47
926	66611	66658	66705	66752	66798	66845	66892	66939	66986	67033	47
927	67080	67127	67173	67220	67267	67314	67361	67408	67454	67501	47
928	67548	67595	67642	67688	67735	67782	67829	67875	67922	67969	47
929	68016	68062	68109	68156	68203	68249	68296	68343	68389	68436	47
930	68483	68530	68576	68623	68670	68716	68763	68810	68856	68903	47
931	68950	68996	69043	69090	69136	69183	69229	69276	69323	69369	47
932	69416	69462	69509	69556	69602	69649	69695	69742	69788	69835	47
933	69882	69928	69975	70021	70068	70114	70161	70207	70254	70300	47
934	70347	70393	70440	70486	70533	70579	70626	70672	70719	70765	47
935	70812	70858	70904	70951	70997	71044	71090	71137	71183	71229	46
936	71276	71322	71369	71415	71461	71508	71554	71600	71647	71693	46
937	71740	71786	71832	71879	71925	71971	72018	72064	72110	72156	46
938	72203	72249	72295	72342	72388	72434	72480	72527	72573	72619	46
939	72666	72712	72758	72804	72851	72897	72943	72989	73035	73082	46
	0	1	2	3	4	5	6	7	8	9	

No. 940—999 Log. [illegible]

No.	0	1	2	3	4	5	6	7	8	9	Diff.
940	73128	73174	73223	73269	73313	73359	73406	73451	[illegible]	[illegible]	46
941	73544	73590	73636	73682	73774	73820	73866	73913	[illegible]	[illegible]	46
942	73961	74007	74143	74189	74235	74281	74327	74373	[illegible]	[illegible]	46
943	74412	74458	74504	74550	74596	74742	74788	74834	[illegible]	[illegible]	46
944	74881	74927	74973	75019	75156	75202	75248	75294	[illegible]	[illegible]	46
945	75342	75388	75434	75570	75616	75661	75707	75753	[illegible]	[illegible]	46
946	75801	75847	75893	75939	76075	76121	76166	76212	[illegible]	[illegible]	46
947	76259	76305	76351	76487	76533	76579	76625	76671	[illegible]	[illegible]	46
948	[illegible]	[illegible]	76990	77036	77081	77127	77173	77219	[illegible]	[illegible]	46
949	77264	77310	77355	77401	77447	77493	77539	77584	[illegible]	[illegible]	46
950	77724	77770	77815	77861	77907	77952	77998	78044	78090	78135	46
951	78181	78226	78272	78317	78363	78409	78454	78500	78546	78591	46
952	78637	78683	78728	78774	78819	78865	78911	78956	79002	79047	46
953	79093	79138	79184	79230	79275	79321	79366	79412	79457	79503	46
954	79548	79594	79639	79685	79730	79776	79821	79867	79912	79958	46
955	80003	80049	80094	80140	80185	80231	80276	80322	80367	80412	46
956	80458	80503	80549	80594	80640	80685	80730	80776	80821	80867	46
957	80912	80957	81003	81048	81093	81139	81184	81229	81275	81320	46
958	81365	81411	81456	81501	81547	81592	81637	81683	81728	81773	46
959	81819	81864	81909	81954	82000	82045	82090	82135	82181	82226	46
960	82271	82316	82362	82407	82452	82497	82543	82588	82633	82678	45
961	82723	82769	82814	82859	82904	82949	82994	83040	83085	83130	45
962	83175	83220	83265	83310	83356	83401	83446	83491	83536	83581	45
963	83626	83671	83716	83762	83807	83852	83897	83942	83987	84032	45
964	84077	84122	84167	84212	84257	84302	84347	84392	84437	84482	45
965	84527	84572	84617	84662	84707	84752	84797	84842	84887	84932	45
966	84977	85022	85067	85112	85157	85202	85247	85292	85337	85382	45
967	85426	85471	85516	85561	85606	85651	85696	85741	85786	85830	45
968	85875	85920	85965	86010	86055	86100	86144	86189	86234	86279	45
969	86324	86369	86413	86458	86503	86548	86593	86637	86682	86727	45
970	86772	86816	86861	86906	86951	86995	87040	87085	87130	87174	45
971	87219	87264	87309	87353	87398	87443	87487	87532	87577	87622	45
972	87666	87711	87756	87800	87845	87890	87934	87979	88024	88068	45
973	88113	88157	88202	88247	88291	88336	88381	88425	88470	88514	45
974	88559	88603	88648	88693	88737	88782	88826	88871	88915	88960	45
975	89005	89049	89094	89138	89183	89227	89272	89316	89361	89405	45
976	89450	89494	89539	89583	89628	89672	89717	89761	89806	89850	44
977	89895	89939	89983	90028	90072	90117	90161	90206	90250	90294	44
978	90339	90383	90428	90472	90516	90561	90605	90650	90694	90738	44
979	90783	90827	90871	90916	90960	91004	91049	91093	91137	91182	44
980	91226	91270	91315	91359	91403	91448	91492	91536	91580	91625	44
981	91669	91713	91757	91802	91846	91890	91934	91979	92023	92067	44
982	92111	92156	92200	92244	92288	92333	92377	92421	92465	92509	44
983	92553	92598	92642	92686	92730	92774	92819	92863	92907	92951	44
984	92995	93039	93083	93127	93172	93216	93260	93304	93348	93392	44
985	93436	93480	93524	93568	93613	93657	93701	93745	93789	93833	44
986	93877	93921	93965	94009	94053	94097	94141	94185	94229	94273	44
987	94317	94361	94405	94449	94493	94537	94581	94625	94669	94713	44
988	94757	94801	94845	94889	94933	94977	95021	95064	95108	95152	44
989	95196	95240	95284	95328	95372	95416	95460	95504	95547	95591	44
990	95635	95679	95723	95767	95811	95854	95898	95942	95986	96030	44
991	96074	96117	96161	96205	96249	96293	96336	96380	96424	96468	44
992	96512	96555	96599	96643	96687	96730	96774	96818	96862	96905	44
993	96949	96993	97037	97080	97124	97168	97212	97255	97299	97343	44
994	97386	97430	97474	97517	97561	97605	97648	97692	97736	97779	44
995	97823	97867	97910	97954	97998	98041	98085	98128	98172	98216	44
996	98259	98303	98346	98390	98434	98477	98521	98564	98608	98652	44
997	98695	98739	98782	98826	98869	98913	98956	99000	99043	99087	44
998	99130	99174	99218	99261	99305	99348	99392	99435	99478	99522	44
999	99565	99609	99652	99696	99739	99783	99826	99870	99913	99957	43
	0	1	2	3	4	5	6	7	8	9	

TABLE

OF THE

PRINCIPLES OF ARITHMETIC.

1. Quantity is every thing which is susceptible of increase and of diminution, number 1.

2. Arithmetic is the science of numbers, number 2.

3. The unit is the term of comparison for all quantities of the same kind, number 4.

4. The number shows how many units or parts of a unit a quantity contains, number 5.

5. An abstract number is that which is not applied to any determined species, number 6.

6. A concrete number is always applied to some kind of thing, number 6.

7. Numeration is the art of expressing and representing numbers, number 7.

8. This numeration is founded upon this principle of convention, that if several figures be ranged upon the same line, the units represented by each of these figures are worth ten times as much as those of the next figure on the right, and consequently only one-tenth of those of the next figure on the left, number 15.

9. Simple numbers only relate to one species of unit, number 18.

10. Compound numbers express quantities the parts of which are compared to different units, number 18.

11. Decimals are parts which decrease from the units in a tenfold proportion; we express them by figures placed on the right of the units, and separated from them by a comma, number 21.

12. A number becomes ten times greater or less as the comma is removed one place towards the right or left, number 28.

13. Addition is an operation by which we express, by a single number, the total value of several numbers of the same kind, number 33.

14. Subtraction is an operation by which we find the remainder, the excess, or the difference of two numbers of the same kind, number 35.

15. Multiplication is an operation by which we repeat a number as many times as there are units in another, number 40.

16. The number that we repeat is called *multiplicand*; that which indicates how many times we repeat it is called *multiplier*; and the result of the multiplication is called *product*, number 41.

17. We call the numbers which we multiply together *factors*, number 42.

Multiplication is a reiterated addition of the multiplicand, as many times as there are units in the multiplier, number 43.

18. The product is always of the same nature as the multiplicand, number 47.

19. In the multiplication of decimal parts, the product ought to have as many decimal figures as there are in both the factors, number 54.

20. Division is an operation by which we seek how many times a number is contained in another, number 59.

21. The number that we divide is called *dividend;* that by which we divide *divisor;* and that which we find *quotient*, number 59.

22. The dividend is always equal to the product of a multiplication of which the divisor and the quotient would be factors, number 59.

23. The nature of the units of the quotient cannot, in general, be determined but by the conditions of the question which gives rise to the division, number 59.

24. The division of decimal parts is reduced to that of whole numbers, in observing to render the number of decimal figures the same in the dividend and divisor, number 68.

25. When the unit is divided into any number of equal parts, we call one one or more of these parts *fraction*, number 78.

26. A fraction is expressed by two numbers, one of which shows into how many parts the unit is divided: it is called *denominator;* the other shows how many of these parts enter into the value of the fraction; we call it *numerator*, number 80.

27. The numerator and the denominator are called the *two terms of the fraction*, number 83.

28. A fractional expression, of which the numerator is greater than the denominator, is greater than a unit, number 84.

29. When a fractional expression is greater than a unit, we find its value in dividing the numerator by the denominator, number 85.

30. We reduce a whole number to a fraction of a determined kind in multiplying it by the denominator of this fraction, number 86.

31. We do not alter the value of a fraction when we multiply or divide its two terms by the same number, numbers 88 and 89.

32. By this principle we explain the reduction of several fractions to the same denominator, and the reduction of fractions to their most simple expression, numbers 90 and 91.

33. A prime number is that which has no other divisor than unity or itself, number 93.

34. A fraction may be considered as the quotient of a division of which the numerator would be the dividend, and the denominator would be the divisor, number 96.

35. To add fractions together, and to subtract the one from the other, they must first be reduced to the same denominator, after which we add or subtract their numerators, and give to the sum or the difference the common denominator, numbers 101 and 102.

36. To multiply a fraction by another fraction, we must multiply numerator by numerator, and denominator by denominator, number 106.

37. To divide a fraction by another, we must multiply it by the divisor fraction inverted, number 109.

38. To estimate a fraction is to seek the value of it in the lower orders of the principal unit of which it makes a part, number 112.

39. A fraction of a fraction, or compound fraction, is equal to the product of all the fractions which enter into its expression, number 114.

40. The addition and the subtraction of compound numbers do not differ from those of simple numbers except by the different subdivisions of the unit, numbers 117 and 118.

41. A number is an aliquot part of another, or is a measure of that other when it is exactly contained in it, number 120.

42. In the multiplication of compound numbers, we consider the lower orders as fractions, with regard to each other, as well as to the principal unit, number 121.

43. In the division of compound numbers, we must always render the divisor simple, number 124 *and fol.*

The square of a number is the product of this number multiplied by itself, number 129.

44. The root of a square is the number which, multiplied by itself, has produced the square, number 130.

45. When a number is not a perfect square, its root is called *surd, irrational,* or *incommensurable,* number 132.

46. The square of a number composed of tens and units contains *the square of the tens, twice the product of the tens by the units, and the square of the units,* number 133.

47. Upon this principle is founded the extraction of the square root of a number composed of more than two figures, number 135 *and fol.*

48. To approach to the square root of a number which is not a perfect square, we place on the right of this number twice as many ciphers as we would have decimals in the root, number 140.

49. To extract the square root of a fraction, we extract the root of the numerator and that of the denominator, if the two terms of the fraction be perfect squares; if not, we reduce the fraction to decimals of an even number of decimal figures, and afterward extract the root, number 142 *and fol.*

50. The cube of a number is the product of this number, multiplied by its square, number 150.

51. The cube root of a cube is the number which, multiplied by its square, produces this cube, number 151.

52. The cube of a number composed of tens and of units contains the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units, number 154.

53. Upon this principle is founded the extraction of the cube root of a number composed of more than three figures, number 155.

54. We approach to the cube root of a number which is not a perfect cube, in placing on the right of this number three times as many ciphers as we would have decimals in the root, number 156.

55. To extract the cube root of a fraction, we extract the cube root of the numerator, and that of the denominator, number 157 *and fol.*

56. The ratio or relation is the result of the comparison of two quantities, number 162.

57. The arithmetical ratio consists in the difference of the two quantities compared, number 163.

58. The geometrical ratio consists in the number of times that one quantity contains another, number 164.

59. An arithmetical ratio is not changed when we add to each of its two terms or when we subtract from each of them the same quantity, number 169.

60. A geometrical ratio is not changed when we multiply or when we divide its two terms by the same number, number 170.

61. Four quantities are in proportion when the ratio of the two first is equal to the ratio of the two last. The proportion is arithmetical or geometrical according to the nature of the ratios which compose it, number 172.

62. The continued proportion is that of which the mean terms are equal to one another, number 174.

63. In every arithmetical proportion, the sum of the extremes is equal to the sum of the means, number 176.

64. In a continued arithmetical proportion, the sum of the extremes is double of the mean term, number 177.

65. In a geometrical proportion, the product of the extremes is equal to the product of the means, number 178.

66. And if the proportion be continued, the product of the extremes is equal to the square of the mean term, number 178.

67. The fourth term of a geometrical proportion is equal to the product of the second and third terms, divided by the first, number 179.

68. If four quantities be such that the product of the extremes is equal to the product of the means, these four quantities are in proportion, number 180.

69. If four quantities be in proportion, they will still be in proportion if we put the extremes in the place of the means, and the means in the place of the extremes; or if we change the places of the extremes, or those of the means, numbers 181 and 182.

70. We can multiply or divide by the same number the two antecedents or the two consequents without disturbing the proportion, number 183.

71. Every change made in a proportion, so that the sum of the antecedent and the consequent, or their difference, may be compared to the antecedent or to the consequent in the same manner in each ratio, will always form a proportion, number 184.

72. The sum or the difference of the antecedents of a proportion is to the sum or the difference of the consequents as one antecedent is to its consequent, number 185.

73. In a series of several equal ratios, the sum of the antecedents is to that of the consequents as one antecedent is to its consequent, number 186.

74. The compound ratio results from two or more ratios, of which we multiply the antecedents together, and the consequents together, number 187.

75. The ratio is double, triple, quadruple, etc. when it is composed of two, three, four, etc. equal ratios, number 189.

76. The products of two or more proportions multiplied in order, that is, term by term, are in proportion, number 190.

77. The squares, the cubes, and in general the similar powers of four quantities in proportion, are also in proportion, number 191.

78. The square roots, cube roots, etc. of four quantities in proportion, are also in proportion, number 192.

79. The Rule of Three has for its object to find a term of a geometrical proportion, the three others being given, number 194.

80. A rule of three is simple when its declaration only contains four terms, of which three are given and one is to find, number 194.

81. A rule of three is direct when the principal quantities are directly proportional to their relatives, number 194.

82. A rule of three is inverse when the principal quantities are reciprocally proportional to their relatives, number 195.

83. A rule of three is compound when its declaration contains more than three known terms; we reduce it to a proportion the ratios of which are compound, number 196.

84. The rule of fellowship has for its object to divide a number into several parts which have given ratios, number 197.

The arithmetical progression is a series of terms which have the same difference, number 204.

85. Any term of an increasing arithmetical progression is composed of the first, plus as many times the ratio or difference as there are terms before it, number 206.

86. A geometrical progression is a series of terms of which each contains the following, or is contained in it the same number of times, number 211.

87. Any term of an increasing geometrical progression is composed of the first multiplied as many times successively by the ratio as there are terms before it, number 212.

88. Logarithms are numbers in arithmetical progression, which answer, term for term, to a like series of numbers in geometrical progression, number 216.

89. In the construction of the logarithms which are at present in use, the arithmetical progression 0, 1, 2, 3, etc. is made to correspond with the decuple geometrical progression 1, 10, 100, 1000, etc., number 218.

90. The index of the logarithm of a number shows in what decade this number is comprised, number 222.

91. The sum of the logarithms of two numbers is equal to the logarithm of their product, number 227.

92. The logarithm of any power of a number is equal to the logarithm of that number, multiplied by the index of the power, number 228.

93. The logarithm of the root of a number is equal to the logarithm of this number, divided by the index or degree of the root, number 230.

94. The logarithm of a whole number joined to a fraction is found in reducing the whole to a fraction, and subtracting the logarithm of the denominator from that of the numerator, number 234.

95. The logarithm of a fraction is the difference of the logarithms of the numerator and denominator, preceded by a sign which shows that this difference is a number which remains yet to be subtracted; so that the logarithms of fractions should be employed according to rules the reverse of those prescribed for the logarithms of whole numbers in performing multiplication and division, numbers 235 and 236.

96. The arithmetical complement of a number is the difference between the number and a unit followed by as many ciphers as there are figures in the number, number 246.

97. By the use of the arithmetical complements, we change the subtractions to additions, and we reduce the logarithms of fractions to the same rules as those which we follow for the logarithms of whole numbers, number 246.

ERRATA.

Page 16, last paragraph but one, for *rules* read *rule*.

Page 37, Example 12, for 819 read 8,19.

Page 59, first paragraph, for *a given number* read *the given number;* and Example 6, for 739÷2956, or 739, etc. read 739÷2956, or $\frac{739}{2956}$, etc. This error is not found in most of the copies.

Page 80, last line, for $9\frac{1}{4}$ read $9\frac{3}{4}$.

Page 81, for $\frac{2\times\dot{3}\times\dot{4}+\dot{5}}{3\times4\times5\times6}$, read $\frac{2\times\dot{3}\times\dot{4}\times\dot{5}}{3\times4\times5\times6}$.

TABLE OF THE SUBJECTS

CONTAINED IN THE ARITHMETIC OF BEZOUT.

	Page
Preface	3
Preliminary notions upon the nature and different kinds of numbers	5
Of numeration and decimals	6
Of the operations of Arithmetic	13
Of the addition of whole numbers and decimal parts	13
Of the subtraction of whole numbers and decimal parts	19
Of multiplication	24
Of multiplication by a number consisting of a single figure	27
Of multiplication by a number consisting of several figures	29
Of the multiplication of decimal parts	31
On some of the uses of multiplication	39
Of the division of whole numbers and decimal parts	40
Of the division of a number consisting of several figures, by a number which has but one	41
Of dividing by a number consisting of many figures	47
Means of abridging the preceding method	50
Of the division of decimal parts	52
Proof of multiplication and division	61
Proof by nine	62
Some uses of the preceding rule	63
Of fractions	67
Of whole numbers considered under the form of fractions	68
Of the changes which we can make in the two terms of a fraction without altering its value	70
Reduction of fractions to the same denominator	71
Reduction of fractions to their most simple expression	73
Different ways of considering a fraction, and consequences thence resulting	77
Of the addition of fractions	78
Of the subtraction of fractions	79
Of the multiplication of fractions	80
Division of fractions	82
Some applications of the preceding rules	83
Of compound numbers	86
Tables of weights, measures, etc. and characters by which the different kinds of units are represented	86 to 91
Addition of compound numbers	91

Page
Subtraction of compound numbers.... 8
Multiplication of compound numbers.... 10
Division of a compound number by a simple number.... 10
Division of a compound number by a compound number.... 10
Of the formation of square numbers and the extraction of their roots.... 10
Of the formation of cube numbers and the extraction of their roots.... 12
Of ratios, proportions, and progressions, and of some rules depending upon them.... 12
Properties of arithmetical proportion.... 13
Properties of geometrical proportion.... 13
Uses of the preceding propositions.... 14
Of the rule of three direct and simple.... 14
Of the rule of three inverse and simple.... 14
Of the compound rule of three.... 14
Of the rule of fellowship.... 14
Model of supply.... 15
Remark on the subject of the preceding rule.... 15
Of some other rules depending upon proportion.... 15
Of the rule of alligation or mixture.... 15
Of arithmetical progression.... 15
Of geometrical progression.... 16
Of logarithms.... 16
Properties of logarithms.... 17
Use of logarithms.... 17
Of numbers the logarithms of which are not found in the table.... 17
Of logarithms the numbers corresponding to which are not found in the table.... 17
Use of the table of arithmetical progression.... 18
Use of the table of geometrical progression for the calculation of compound interest.... 18
Of annuities.... 18
Conjoined proportion.... 18
Rule of a false position.... 19
Rule of two false positions.... 20
Rule of change.... 20
Rule of discount.... 20
Rule of alligation of the first kind.... 20
Rule of alligation of the second kind.... 20
Of barter.... 20
Of duodecimals.... 21
Table of logarithms from 1 as far as 10000.... 213 to 21
Table of the principles of arithmetic.... 21

FINIS.

MAY 8 1944